Hyperbolic Geometry

Release 10.6

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Jun 27, 2025

CONTENTS

CHAPTER

ONE

HYPERBOLIC POINTS

This module implements points in hyperbolic space of arbitrary dimension. It also contains the implementations for specific models of hyperbolic geometry.

This module also implements ideal points in hyperbolic space of arbitrary dimension. It also contains the implementations for specific models of hyperbolic geometry.

Note that not all models of hyperbolic space are bounded, meaning that the ideal boundary is not the topological boundary of the set underlying tho model. For example, the unit disk model is bounded with boundary given by the unit sphere. The hyperboloid model is not bounded.

AUTHORS:

- Greg Laun (2013): initial version

EXAMPLES:

We can construct points in the upper half plane model, abbreviated UHP for convenience:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_point(2 + I)
Point in UHP I + 2
sage: g = UHP.get_point(3 + I)
sage: g.dist(UHP.get_point(I))
arccosh(11/2)
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.get_point(Integer(2) + I)
Point in UHP I + 2
>>> g = UHP.get_point(Integer(3) + I)
>>> g.dist(UHP.get_point(I))
arccosh(11/2)
```

We can also construct boundary points in the upper half plane model:

```
sage: UHP.get_point(3)
Boundary point in UHP 3
```

```
>>> from sage.all import *
>>> UHP.get_point(Integer(3))
Boundary point in UHP 3
```

Some more examples:

```
sage: HyperbolicPlane().UHP().get_point(0)
Boundary point in UHP 0

sage: HyperbolicPlane().PD().get_point(I/2)
Point in PD 1/2*I

sage: HyperbolicPlane().KM().get_point((0,1))
Boundary point in KM (0, 1)

sage: HyperbolicPlane().HM().get_point((0,0,1))
Point in HM (0, 0, 1)
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_point(Integer(0))
Boundary point in UHP 0

>>> HyperbolicPlane().PD().get_point(I/Integer(2))
Point in PD 1/2*I

>>> HyperbolicPlane().KM().get_point((Integer(0),Integer(1)))
Boundary point in KM (0, 1)

>>> HyperbolicPlane().HM().get_point((Integer(0),Integer(0),Integer(1)))
Point in HM (0, 0, 1)
```

class sage.geometry.hyperbolic_space.hyperbolic_point.**HyperbolicPoint**(*model*, *coordinates*, *is_boundary*, *check=True*, ***graphics_options*)

Bases: `Element`

Abstract base class for hyperbolic points. This class should never be instantiated.

INPUT:

- `model` – the model of the hyperbolic space
- `coordinates` – the coordinates of a hyperbolic point in the appropriate model
- `is_boundary` – whether the point is a boundary point
- `check` – boolean (default: `True`); if `True`, then check to make sure the coordinates give a valid point in the model

EXAMPLES:

Comparison between different models is performed via coercion:

```
sage: UHP = HyperbolicPlane().UHP()
sage: p = UHP.get_point(.2 + .3*I); p
Point in UHP 0.200000000000000 + 0.300000000000000*I

sage: PD = HyperbolicPlane().PD()
sage: q = PD.get_point(0.2 + 0.3*I); q
Point in PD 0.200000000000000 + 0.300000000000000*I
```

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```
sage: p == q
False
sage: PD(p)
Point in PD 0.231213872832370 - 0.502890173410405*I

sage: bool(p.coordinates() == q.coordinates())
True
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> p = UHP.get_point(RealNumber('.2') + RealNumber('.3')*I); p
Point in UHP 0.200000000000000 + 0.300000000000000*I

>>> PD = HyperbolicPlane().PD()
>>> q = PD.get_point(RealNumber('0.2') + RealNumber('0.3')*I); q
Point in PD 0.200000000000000 + 0.300000000000000*I

>>> p == q
False
>>> PD(p)
Point in PD 0.231213872832370 - 0.502890173410405*I

>>> bool(p.coordinates() == q.coordinates())
True
```

Similarly for boundary points:

```
sage: p = UHP.get_point(-1); p
Boundary point in UHP -1

sage: q = PD.get_point(-1); q
Boundary point in PD -1

sage: p == q
True
sage: PD(p)
Boundary point in PD -1
```

```
>>> from sage.all import *
>>> p = UHP.get_point(-Integer(1)); p
Boundary point in UHP -1

>>> q = PD.get_point(-Integer(1)); q
Boundary point in PD -1

>>> p == q
True
>>> PD(p)
Boundary point in PD -1
```

It is an error to specify a point that does not lie in the appropriate model:

```
sage: HyperbolicPlane().UHP().get_point(0.2 - 0.3*I)
Traceback (most recent call last):
...
ValueError: 0.200000000000000 - 0.300000000000000*I is not a valid point in the␣
↪UHP model

sage: HyperbolicPlane().PD().get_point(1.2)
Traceback (most recent call last):
...
ValueError: 1.20000000000000 is not a valid point in the PD model

sage: HyperbolicPlane().KM().get_point((1,1))
Traceback (most recent call last):
...
ValueError: (1, 1) is not a valid point in the KM model

sage: HyperbolicPlane().HM().get_point((1, 1, 1))
Traceback (most recent call last):
...
ValueError: (1, 1, 1) is not a valid point in the HM model
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_point(RealNumber('0.2') - RealNumber('0.3')*I)
Traceback (most recent call last):
...
ValueError: 0.200000000000000 - 0.300000000000000*I is not a valid point in the␣
↪UHP model

>>> HyperbolicPlane().PD().get_point(RealNumber('1.2'))
Traceback (most recent call last):
...
ValueError: 1.20000000000000 is not a valid point in the PD model

>>> HyperbolicPlane().KM().get_point((Integer(1),Integer(1)))
Traceback (most recent call last):
...
ValueError: (1, 1) is not a valid point in the KM model

>>> HyperbolicPlane().HM().get_point((Integer(1), Integer(1), Integer(1)))
Traceback (most recent call last):
...
ValueError: (1, 1, 1) is not a valid point in the HM model
```

It is an error to specify an interior point of hyperbolic space as a boundary point:

```
sage: HyperbolicPlane().UHP().get_point(0.2 + 0.3*I, is_boundary=True)
Traceback (most recent call last):
...
ValueError: 0.200000000000000 + 0.300000000000000*I is not a valid boundary point␣
↪in the UHP model
```

```
>>> from sage.all import *
```

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```
>>> HyperbolicPlane().UHP().get_point(RealNumber('0.2') + RealNumber('0.3')*I, is_
↪boundary=True)
Traceback (most recent call last):
...
ValueError: 0.200000000000000 + 0.300000000000000*I is not a valid boundary point␣
↪in the UHP model
```

coordinates()

Return the coordinates of the point.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_point(2 + I).coordinates()
I + 2

sage: HyperbolicPlane().PD().get_point(1/2 + 1/2*I).coordinates()
1/2*I + 1/2

sage: HyperbolicPlane().KM().get_point((1/3, 1/4)).coordinates()
(1/3, 1/4)

sage: HyperbolicPlane().HM().get_point((0,0,1)).coordinates()
(0, 0, 1)
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_point(Integer(2) + I).coordinates()
I + 2

>>> HyperbolicPlane().PD().get_point(Integer(1)/Integer(2) + Integer(1)/
↪Integer(2)*I).coordinates()
1/2*I + 1/2

>>> HyperbolicPlane().KM().get_point((Integer(1)/Integer(3), Integer(1)/
↪Integer(4))).coordinates()
(1/3, 1/4)

>>> HyperbolicPlane().HM().get_point((Integer(0),Integer(0),Integer(1))).
↪coordinates()
(0, 0, 1)
```

graphics_options()

Return the graphics options of the current point.

EXAMPLES:

```
sage: p = HyperbolicPlane().UHP().get_point(2 + I, color='red')
sage: p.graphics_options()
{'color': 'red'}
```

```
>>> from sage.all import *
>>> p = HyperbolicPlane().UHP().get_point(Integer(2) + I, color='red')
>>> p.graphics_options()
{'color': 'red'}
```

is_boundary()

Return `True` if `self` is a boundary point.

EXAMPLES:

```
sage: PD = HyperbolicPlane().PD()
sage: p = PD.get_point(0.5+.2*I)
sage: p.is_boundary()
False
sage: p = PD.get_point(I)
sage: p.is_boundary()
True
```

```
>>> from sage.all import *
>>> PD = HyperbolicPlane().PD()
>>> p = PD.get_point(RealNumber('0.5')+RealNumber('.2')*I)
>>> p.is_boundary()
False
>>> p = PD.get_point(I)
>>> p.is_boundary()
True
```

model()

Return the model to which the *HyperbolicPoint* belongs.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_point(I).model()
Hyperbolic plane in the Upper Half Plane Model

sage: HyperbolicPlane().PD().get_point(0).model()
Hyperbolic plane in the Poincare Disk Model

sage: HyperbolicPlane().KM().get_point((0,0)).model()
Hyperbolic plane in the Klein Disk Model

sage: HyperbolicPlane().HM().get_point((0,0,1)).model()
Hyperbolic plane in the Hyperboloid Model
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_point(I).model()
Hyperbolic plane in the Upper Half Plane Model

>>> HyperbolicPlane().PD().get_point(Integer(0)).model()
Hyperbolic plane in the Poincare Disk Model

>>> HyperbolicPlane().KM().get_point((Integer(0),Integer(0))).model()
Hyperbolic plane in the Klein Disk Model

>>> HyperbolicPlane().HM().get_point((Integer(0),Integer(0),Integer(1))).
↪model()
Hyperbolic plane in the Hyperboloid Model
```

show(*boundary=True*, ***options*)

Plot `self`.

EXAMPLES:

```
sage: HyperbolicPlane().PD().get_point(0).show()                              #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
sage: HyperbolicPlane().KM().get_point((0,0)).show()                          #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
sage: HyperbolicPlane().HM().get_point((0,0,1)).show()                        #␣
↪needs sage.plot
Graphics3d Object
```

```
>>> from sage.all import *
>>> HyperbolicPlane().PD().get_point(Integer(0)).show()                       ␣
↪     # needs sage.plot
Graphics object consisting of 2 graphics primitives
>>> HyperbolicPlane().KM().get_point((Integer(0),Integer(0))).show()          ␣
↪              # needs sage.plot
Graphics object consisting of 2 graphics primitives
>>> HyperbolicPlane().HM().get_point((Integer(0),Integer(0),Integer(1))).
↪show()                       # needs sage.plot
Graphics3d Object
```

symmetry_involution()

Return the involutory isometry fixing the given point.

EXAMPLES:

```
sage: z = HyperbolicPlane().UHP().get_point(3 + 2*I)
sage: z.symmetry_involution()
Isometry in UHP
[  3/2 -13/2]
[  1/2  -3/2]

sage: HyperbolicPlane().UHP().get_point(I).symmetry_involution()
Isometry in UHP
[ 0 -1]
[ 1  0]

sage: HyperbolicPlane().PD().get_point(0).symmetry_involution()
Isometry in PD
[-I  0]
[ 0  I]

sage: HyperbolicPlane().KM().get_point((0, 0)).symmetry_involution()
Isometry in KM
[-1  0  0]
[ 0 -1  0]
[ 0  0  1]

sage: HyperbolicPlane().HM().get_point((0,0,1)).symmetry_involution()
```

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```
Isometry in HM
[-1  0  0]
[ 0 -1  0]
[ 0  0  1]

sage: p = HyperbolicPlane().UHP().random_element()
sage: A = p.symmetry_involution()
sage: p.dist(A*p)  # abs tol 1e-10
0

sage: A.preserves_orientation()
True

sage: A*A == HyperbolicPlane().UHP().get_isometry(identity_matrix(2))        #␣
↪needs scipy
True
```

```
>>> from sage.all import *
>>> z = HyperbolicPlane().UHP().get_point(Integer(3) + Integer(2)*I)
>>> z.symmetry_involution()
Isometry in UHP
[  3/2 -13/2]
[  1/2  -3/2]

>>> HyperbolicPlane().UHP().get_point(I).symmetry_involution()
Isometry in UHP
[ 0 -1]
[ 1  0]

>>> HyperbolicPlane().PD().get_point(Integer(0)).symmetry_involution()
Isometry in PD
[-I  0]
[ 0  I]

>>> HyperbolicPlane().KM().get_point((Integer(0), Integer(0))).symmetry_
↪involution()
Isometry in KM
[-1  0  0]
[ 0 -1  0]
[ 0  0  1]

>>> HyperbolicPlane().HM().get_point((Integer(0),Integer(0),Integer(1))).
↪symmetry_involution()
Isometry in HM
[-1  0  0]
[ 0 -1  0]
[ 0  0  1]

>>> p = HyperbolicPlane().UHP().random_element()
>>> A = p.symmetry_involution()
>>> p.dist(A*p)  # abs tol 1e-10
0
```

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```
>>> A.preserves_orientation()
True

>>> A*A == HyperbolicPlane().UHP().get_isometry(identity_matrix(Integer(2))) ␣
↪     # needs scipy
True
```

to_model(*model*)

Convert `self` to the `model`.

INPUT:

- `other` – (a string representing) the image model

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: PD = HyperbolicPlane().PD()
sage: PD.get_point(1/2+I/2).to_model(UHP)
Point in UHP I + 2
sage: PD.get_point(1/2+I/2).to_model('UHP')
Point in UHP I + 2
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> PD = HyperbolicPlane().PD()
>>> PD.get_point(Integer(1)/Integer(2)+I/Integer(2)).to_model(UHP)
Point in UHP I + 2
>>> PD.get_point(Integer(1)/Integer(2)+I/Integer(2)).to_model('UHP')
Point in UHP I + 2
```

update_graphics(*update=False*, ***options*)

Update the graphics options of a *HyperbolicPoint*. If `update` is `True`, update rather than overwrite.

EXAMPLES:

```
sage: p = HyperbolicPlane().UHP().get_point(I); p.graphics_options()
{}

sage: p.update_graphics(color = "red"); p.graphics_options()
{'color': 'red'}

sage: p.update_graphics(color = "blue"); p.graphics_options()
{'color': 'blue'}

sage: p.update_graphics(True, size = 20); p.graphics_options()
{'color': 'blue', 'size': 20}
```

```
>>> from sage.all import *
>>> p = HyperbolicPlane().UHP().get_point(I); p.graphics_options()
{}
```

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```
>>> p.update_graphics(color = "red"); p.graphics_options()
{'color': 'red'}

>>> p.update_graphics(color = "blue"); p.graphics_options()
{'color': 'blue'}

>>> p.update_graphics(True, size = Integer(20)); p.graphics_options()
{'color': 'blue', 'size': 20}
```

class sage.geometry.hyperbolic_space.hyperbolic_point.**HyperbolicPointUHP**(*model*, *coordinates*, *is_boundary*, *check=True*, ***graphics_options*)

Bases: *HyperbolicPoint*

A point in the UHP model.

INPUT:

- the coordinates of a point in the unit disk in the complex plane **C**

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_point(2*I)
Point in UHP 2*I

sage: HyperbolicPlane().UHP().get_point(1)
Boundary point in UHP 1
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_point(Integer(2)*I)
Point in UHP 2*I

>>> HyperbolicPlane().UHP().get_point(Integer(1))
Boundary point in UHP 1
```

show(*boundary=True*, ***options*)

Plot `self`.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_point(I).show()
Graphics object consisting of 2 graphics primitives
sage: HyperbolicPlane().UHP().get_point(0).show()                              #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
sage: HyperbolicPlane().UHP().get_point(infinity).show()
Traceback (most recent call last):
...
NotImplementedError: can...t draw the point infinity
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_point(I).show()
Graphics object consisting of 2 graphics primitives
>>> HyperbolicPlane().UHP().get_point(Integer(0)).show()
↪     # needs sage.plot
Graphics object consisting of 2 graphics primitives
>>> HyperbolicPlane().UHP().get_point(infinity).show()
Traceback (most recent call last):
...
NotImplementedError: can...t draw the point infinity
```

symmetry_involution()

Return the involutory isometry fixing the given point.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_point(3 + 2*I).symmetry_involution()
Isometry in UHP
[  3/2 -13/2]
[  1/2  -3/2]
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_point(Integer(3) + Integer(2)*I).symmetry_
↪involution()
Isometry in UHP
[  3/2 -13/2]
[  1/2  -3/2]
```

CHAPTER

TWO

HYPERBOLIC ISOMETRIES

This module implements the abstract base class for isometries of hyperbolic space of arbitrary dimension. It also contains the implementations for specific models of hyperbolic geometry.

The isometry groups of all implemented models are either matrix Lie groups or are doubly covered by matrix Lie groups. As such, the isometry constructor takes a matrix as input. However, since the isometries themselves may not be matrices, quantities like the trace and determinant are not directly accessible from this class.

AUTHORS:

- Greg Laun (2013): initial version

EXAMPLES:

We can construct isometries in the upper half plane model, abbreviated UHP for convenience:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_isometry(matrix(2,[1,2,3,4]))
Isometry in UHP
[1 2]
[3 4]
sage: A = UHP.get_isometry(matrix(2,[0,1,1,0]))
sage: A.inverse()
Isometry in UHP
[0 1]
[1 0]
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.get_isometry(matrix(Integer(2),[Integer(1),Integer(2),Integer(3),Integer(4)]))
Isometry in UHP
[1 2]
[3 4]
>>> A = UHP.get_isometry(matrix(Integer(2),[Integer(0),Integer(1),Integer(1),
↪Integer(0)]))
>>> A.inverse()
Isometry in UHP
[0 1]
[1 0]
```

class sage.geometry.hyperbolic_space.hyperbolic_isometry.**HyperbolicIsometry**(*model*, *A*, *check=True*)

Bases: Morphism

Abstract base class for hyperbolic isometries. This class should never be instantiated.

INPUT:

- A – a matrix representing a hyperbolic isometry in the appropriate model

EXAMPLES:

```
sage: HyperbolicPlane().HM().get_isometry(identity_matrix(3))
Isometry in HM
[1 0 0]
[0 1 0]
[0 0 1]
```

```
>>> from sage.all import *
>>> HyperbolicPlane().HM().get_isometry(identity_matrix(Integer(3)))
Isometry in HM
[1 0 0]
[0 1 0]
[0 0 1]
```

attracting_fixed_point()

For a hyperbolic isometry, return the attracting fixed point; otherwise raise a ValueError.

OUTPUT: a hyperbolic point

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: A = UHP.get_isometry(Matrix(2,[4,0,0,1/4]))
sage: A.attracting_fixed_point()
Boundary point in UHP +Infinity
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> A = UHP.get_isometry(Matrix(Integer(2),[Integer(4),Integer(0),Integer(0),
↪Integer(1)/Integer(4)]))
>>> A.attracting_fixed_point()
Boundary point in UHP +Infinity
```

axis()

For a hyperbolic isometry, return the axis of the transformation; otherwise raise a ValueError.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: H = UHP.get_isometry(matrix(2,[2,0,0,1/2]))
sage: H.axis()
Geodesic in UHP from 0 to +Infinity
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> H = UHP.get_isometry(matrix(Integer(2),[Integer(2),Integer(0),Integer(0),
↪Integer(1)/Integer(2)]))
>>> H.axis()
Geodesic in UHP from 0 to +Infinity
```

It is an error to call this function on an isometry that is not hyperbolic:

```
sage: P = UHP.get_isometry(matrix(2,[1,4,0,1]))
sage: P.axis()
Traceback (most recent call last):
...
ValueError: the isometry is not hyperbolic: axis is undefined
```

```
>>> from sage.all import *
>>> P = UHP.get_isometry(matrix(Integer(2),[Integer(1),Integer(4),Integer(0),
↪Integer(1)]))
>>> P.axis()
Traceback (most recent call last):
...
ValueError: the isometry is not hyperbolic: axis is undefined
```

classification()

Classify the hyperbolic isometry as elliptic, parabolic, hyperbolic or a reflection.

A hyperbolic isometry fixes two points on the boundary of hyperbolic space, a parabolic isometry fixes one point on the boundary of hyperbolic space, and an elliptic isometry fixes no points.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: H = UHP.get_isometry(matrix(2,[2,0,0,1/2]))
sage: H.classification()
'hyperbolic'

sage: P = UHP.get_isometry(matrix(2,[1,1,0,1]))
sage: P.classification()
'parabolic'

sage: E = UHP.get_isometry(matrix(2,[-1,0,0,1]))
sage: E.classification()
'reflection'
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> H = UHP.get_isometry(matrix(Integer(2),[Integer(2),Integer(0),Integer(0),
↪Integer(1)/Integer(2)]))
>>> H.classification()
'hyperbolic'

>>> P = UHP.get_isometry(matrix(Integer(2),[Integer(1),Integer(1),Integer(0),
↪Integer(1)]))
>>> P.classification()
'parabolic'

>>> E = UHP.get_isometry(matrix(Integer(2),[-Integer(1),Integer(0),Integer(0),
↪Integer(1)]))
>>> E.classification()
'reflection'
```

fixed_geodesic()

If `self` is a reflection in a geodesic, return that geodesic.

EXAMPLES:

```
sage: A = HyperbolicPlane().PD().get_isometry(matrix([[0, 1], [1, 0]]))
sage: A.fixed_geodesic()
Geodesic in PD from -1 to 1
```

```
>>> from sage.all import *
>>> A = HyperbolicPlane().PD().get_isometry(matrix([[Integer(0), Integer(1)],␣
↪[Integer(1), Integer(0)]]))
>>> A.fixed_geodesic()
Geodesic in PD from -1 to 1
```

fixed_point_set()

Return a list containing the fixed point set of orientation-preserving isometries.

OUTPUT: list of hyperbolic points or a hyperbolic geodesic

EXAMPLES:

```
sage: KM = HyperbolicPlane().KM()
sage: H = KM.get_isometry(matrix([[5/3,0,4/3], [0,1,0], [4/3,0,5/3]]))
sage: g = H.fixed_point_set(); g
Geodesic in KM from (1, 0) to (-1, 0)
sage: H(g.start()) == g.start()
True
sage: H(g.end()) == g.end()
True
sage: A = KM.get_isometry(matrix([[1,0,0], [0,-1,0], [0,0,1]]))
sage: A.preserves_orientation()
False
sage: A.fixed_point_set()
Geodesic in KM from (1, 0) to (-1, 0)
```

```
>>> from sage.all import *
>>> KM = HyperbolicPlane().KM()
>>> H = KM.get_isometry(matrix([[Integer(5)/Integer(3),Integer(0),Integer(4)/
↪Integer(3)], [Integer(0),Integer(1),Integer(0)], [Integer(4)/Integer(3),
↪Integer(0),Integer(5)/Integer(3)]]))
>>> g = H.fixed_point_set(); g
Geodesic in KM from (1, 0) to (-1, 0)
>>> H(g.start()) == g.start()
True
>>> H(g.end()) == g.end()
True
>>> A = KM.get_isometry(matrix([[Integer(1),Integer(0),Integer(0)],␣
↪[Integer(0),-Integer(1),Integer(0)], [Integer(0),Integer(0),Integer(1)]]))
>>> A.preserves_orientation()
False
>>> A.fixed_point_set()
Geodesic in KM from (1, 0) to (-1, 0)
```

```
sage: B = KM.get_isometry(identity_matrix(3))
sage: B.fixed_point_set()
Traceback (most recent call last):
```

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```
...
ValueError: the identity transformation fixes the entire hyperbolic plane
```

```
>>> from sage.all import *
>>> B = KM.get_isometry(identity_matrix(Integer(3)))
>>> B.fixed_point_set()
Traceback (most recent call last):
...
ValueError: the identity transformation fixes the entire hyperbolic plane
```

is_identity()

Return `True` if `self` is the identity isometry.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_isometry(matrix(2,[4,1,3,2])).is_identity()
False
sage: UHP.get_isometry(identity_matrix(2)).is_identity()
True
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.get_isometry(matrix(Integer(2),[Integer(4),Integer(1),Integer(3),
↪Integer(2)])).is_identity()
False
>>> UHP.get_isometry(identity_matrix(Integer(2))).is_identity()
True
```

matrix()

Return the matrix of the isometry.

Note

We do not allow the `matrix` constructor to work as these may be elements of a projective group (ex. $PSL(n, \mathbf{R})$), so these isometries aren't true matrices.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_isometry(-identity_matrix(2)).matrix()
[-1  0]
[ 0 -1]
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.get_isometry(-identity_matrix(Integer(2))).matrix()
[-1  0]
[ 0 -1]
```

model()

Return the model to which `self` belongs.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_isometry(identity_matrix(2)).model()
Hyperbolic plane in the Upper Half Plane Model

sage: HyperbolicPlane().PD().get_isometry(identity_matrix(2)).model()
Hyperbolic plane in the Poincare Disk Model

sage: HyperbolicPlane().KM().get_isometry(identity_matrix(3)).model()
Hyperbolic plane in the Klein Disk Model

sage: HyperbolicPlane().HM().get_isometry(identity_matrix(3)).model()
Hyperbolic plane in the Hyperboloid Model
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_isometry(identity_matrix(Integer(2))).model()
Hyperbolic plane in the Upper Half Plane Model

>>> HyperbolicPlane().PD().get_isometry(identity_matrix(Integer(2))).model()
Hyperbolic plane in the Poincare Disk Model

>>> HyperbolicPlane().KM().get_isometry(identity_matrix(Integer(3))).model()
Hyperbolic plane in the Klein Disk Model

>>> HyperbolicPlane().HM().get_isometry(identity_matrix(Integer(3))).model()
Hyperbolic plane in the Hyperboloid Model
```

preserves_orientation()

Return `True` if `self` is orientation-preserving and `False` otherwise.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: A = UHP.get_isometry(identity_matrix(2))
sage: A.preserves_orientation()
True
sage: B = UHP.get_isometry(matrix(2,[0,1,1,0]))
sage: B.preserves_orientation()
False
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> A = UHP.get_isometry(identity_matrix(Integer(2)))
>>> A.preserves_orientation()
True
>>> B = UHP.get_isometry(matrix(Integer(2),[Integer(0),Integer(1),Integer(1),
↪Integer(0)]))
>>> B.preserves_orientation()
False
```

repelling_fixed_point()

For a hyperbolic isometry, return the attracting fixed point; otherwise raise a `ValueError`.

OUTPUT: a hyperbolic point

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: A = UHP.get_isometry(Matrix(2,[4,0,0,1/4]))
sage: A.repelling_fixed_point()
Boundary point in UHP 0
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> A = UHP.get_isometry(Matrix(Integer(2),[Integer(4),Integer(0),Integer(0),
↪Integer(1)/Integer(4)]))
>>> A.repelling_fixed_point()
Boundary point in UHP 0
```

to_model(*other*)

Convert the current object to image in another model.

INPUT:

- `other` – (a string representing) the image model

EXAMPLES:

```
sage: H = HyperbolicPlane()
sage: UHP = H.UHP()
sage: PD = H.PD()
sage: KM = H.KM()
sage: HM = H.HM()

sage: A = UHP.get_isometry(identity_matrix(2))
sage: A.to_model(HM)
Isometry in HM
[1 0 0]
[0 1 0]
[0 0 1]
sage: A.to_model('HM')
Isometry in HM
[1 0 0]
[0 1 0]
[0 0 1]

sage: A = PD.get_isometry(matrix([[I, 0], [0, -I]]))
sage: A.to_model(UHP)
Isometry in UHP
[ 0  1]
[-1  0]
sage: A.to_model(HM)
Isometry in HM
[-1  0  0]
[ 0 -1  0]
[ 0  0  1]
```

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```
sage: A.to_model(KM)
Isometry in KM
[-1  0  0]
[ 0 -1  0]
[ 0  0  1]

sage: A = HM.get_isometry(diagonal_matrix([-1, -1, 1]))
sage: A.to_model('UHP')
Isometry in UHP
[ 0 -1]
[ 1  0]
sage: A.to_model('PD')
Isometry in PD
[-I  0]
[ 0  I]
sage: A.to_model('KM')
Isometry in KM
[-1  0  0]
[ 0 -1  0]
[ 0  0  1]
```

```
>>> from sage.all import *
>>> H = HyperbolicPlane()
>>> UHP = H.UHP()
>>> PD = H.PD()
>>> KM = H.KM()
>>> HM = H.HM()

>>> A = UHP.get_isometry(identity_matrix(Integer(2)))
>>> A.to_model(HM)
Isometry in HM
[1 0 0]
[0 1 0]
[0 0 1]
>>> A.to_model('HM')
Isometry in HM
[1 0 0]
[0 1 0]
[0 0 1]

>>> A = PD.get_isometry(matrix([[I, Integer(0)], [Integer(0), -I]]))
>>> A.to_model(UHP)
Isometry in UHP
[ 0  1]
[-1  0]
>>> A.to_model(HM)
Isometry in HM
[-1  0  0]
[ 0 -1  0]
[ 0  0  1]
>>> A.to_model(KM)
Isometry in KM
```

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```
[-1  0  0]
[ 0 -1  0]
[ 0  0  1]

>>> A = HM.get_isometry(diagonal_matrix([-Integer(1), -Integer(1),␣
↪Integer(1)]))
>>> A.to_model('UHP')
Isometry in UHP
[ 0 -1]
[ 1  0]
>>> A.to_model('PD')
Isometry in PD
[-I  0]
[ 0  I]
>>> A.to_model('KM')
Isometry in KM
[-1  0  0]
[ 0 -1  0]
[ 0  0  1]
```

translation_length()

For hyperbolic elements, return the translation length; otherwise, raise a ValueError.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: H = UHP.get_isometry(matrix(2,[2,0,0,1/2]))
sage: H.translation_length()
2*arccosh(5/4)
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> H = UHP.get_isometry(matrix(Integer(2),[Integer(2),Integer(0),Integer(0),
↪Integer(1)/Integer(2)]))
>>> H.translation_length()
2*arccosh(5/4)
```

```
sage: f_1 = UHP.get_point(-1)
sage: f_2 = UHP.get_point(1)
sage: H = UHP.isometry_from_fixed_points(f_1, f_2)
sage: p = UHP.get_point(exp(i*7*pi/8))
sage: bool((p.dist(H*p) - H.translation_length()) < 10**-9)
True
```

```
>>> from sage.all import *
>>> f_1 = UHP.get_point(-Integer(1))
>>> f_2 = UHP.get_point(Integer(1))
>>> H = UHP.isometry_from_fixed_points(f_1, f_2)
>>> p = UHP.get_point(exp(i*Integer(7)*pi/Integer(8)))
>>> bool((p.dist(H*p) - H.translation_length()) < Integer(10)**-Integer(9))
True
```

class sage.geometry.hyperbolic_space.hyperbolic_isometry.**HyperbolicIsometryKM**(*model*, *A*, *check=True*)

Bases: *HyperbolicIsometry*

Create a hyperbolic isometry in the KM model.

INPUT:

- a matrix in $SO(2,1)$

EXAMPLES:

```
sage: HyperbolicPlane().KM().get_isometry(identity_matrix(3))
Isometry in KM
[1 0 0]
[0 1 0]
[0 0 1]
```

```
>>> from sage.all import *
>>> HyperbolicPlane().KM().get_isometry(identity_matrix(Integer(3)))
Isometry in KM
[1 0 0]
[0 1 0]
[0 0 1]
```

class sage.geometry.hyperbolic_space.hyperbolic_isometry.**HyperbolicIsometryPD**(*model*, *A*, *check=True*)

Bases: *HyperbolicIsometry*

Create a hyperbolic isometry in the PD model.

INPUT:

- a matrix in $PU(1,1)$

EXAMPLES:

```
sage: HyperbolicPlane().PD().get_isometry(identity_matrix(2))
Isometry in PD
[1 0]
[0 1]
```

```
>>> from sage.all import *
>>> HyperbolicPlane().PD().get_isometry(identity_matrix(Integer(2)))
Isometry in PD
[1 0]
[0 1]
```

preserves_orientation()

Return `True` if `self` preserves orientation and `False` otherwise.

EXAMPLES:

```
sage: PD = HyperbolicPlane().PD()
sage: PD.get_isometry(matrix([[-I, 0], [0, I]])).preserves_orientation()
True
```

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```
sage: PD.get_isometry(matrix([[0, I], [I, 0]])).preserves_orientation()
False
```

```
>>> from sage.all import *
>>> PD = HyperbolicPlane().PD()
>>> PD.get_isometry(matrix([[-I, Integer(0)], [Integer(0), I]])).preserves_
↪orientation()
True
>>> PD.get_isometry(matrix([[Integer(0), I], [I, Integer(0)]])).preserves_
↪orientation()
False
```

class sage.geometry.hyperbolic_space.hyperbolic_isometry.**HyperbolicIsometryUHP**(*model*, *A*, *check=True*)

Bases: *HyperbolicIsometry*

Create a hyperbolic isometry in the UHP model.

INPUT:

- a matrix in $GL(2, \mathbf{R})$

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_isometry(identity_matrix(2))
Isometry in UHP
[1 0]
[0 1]
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_isometry(identity_matrix(Integer(2)))
Isometry in UHP
[1 0]
[0 1]
```

attracting_fixed_point()

Return the attracting fixed point.

Otherwise, this raises a ValueError.

OUTPUT: a hyperbolic point

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: A = matrix(2,[4,0,0,1/4])
sage: UHP.get_isometry(A).attracting_fixed_point()
Boundary point in UHP +Infinity
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> A = matrix(Integer(2),[Integer(4),Integer(0),Integer(0),Integer(1)/
↪Integer(4)])
>>> UHP.get_isometry(A).attracting_fixed_point()
Boundary point in UHP +Infinity
```

classification()

Classify the hyperbolic isometry as elliptic, parabolic, or hyperbolic.

A hyperbolic isometry fixes two points on the boundary of hyperbolic space, a parabolic isometry fixes one point on the boundary of hyperbolic space, and an elliptic isometry fixes no points.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_isometry(identity_matrix(2)).classification()
'identity'

sage: UHP.get_isometry(4*identity_matrix(2)).classification()
'identity'

sage: UHP.get_isometry(matrix(2,[2,0,0,1/2])).classification()
'hyperbolic'

sage: UHP.get_isometry(matrix(2, [0, 3, -1/3, 6])).classification()
'hyperbolic'

sage: UHP.get_isometry(matrix(2,[1,1,0,1])).classification()
'parabolic'

sage: UHP.get_isometry(matrix(2,[-1,0,0,1])).classification()
'reflection'
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.get_isometry(identity_matrix(Integer(2))).classification()
'identity'

>>> UHP.get_isometry(Integer(4)*identity_matrix(Integer(2))).classification()
'identity'

>>> UHP.get_isometry(matrix(Integer(2),[Integer(2),Integer(0),Integer(0),
↪Integer(1)/Integer(2)])).classification()
'hyperbolic'

>>> UHP.get_isometry(matrix(Integer(2), [Integer(0), Integer(3), -Integer(1)/
↪Integer(3), Integer(6)])).classification()
'hyperbolic'

>>> UHP.get_isometry(matrix(Integer(2),[Integer(1),Integer(1),Integer(0),
↪Integer(1)])).classification()
'parabolic'

>>> UHP.get_isometry(matrix(Integer(2),[-Integer(1),Integer(0),Integer(0),
↪Integer(1)])).classification()
'reflection'
```

fixed_point_set()

Return a list or geodesic containing the fixed point set of orientation-preserving isometries.

OUTPUT: list of hyperbolic points or a hyperbolic geodesic

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: H = UHP.get_isometry(matrix(2, [-2/3,-1/3,-1/3,-2/3]))
sage: g = H.fixed_point_set(); g
Geodesic in UHP from -1 to 1
sage: H(g.start()) == g.start()
True
sage: H(g.end()) == g.end()
True
sage: A = UHP.get_isometry(matrix(2,[0,1,1,0]))
sage: A.preserves_orientation()
False
sage: A.fixed_point_set()
Geodesic in UHP from 1 to -1
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> H = UHP.get_isometry(matrix(Integer(2), [-Integer(2)/Integer(3),-
↪Integer(1)/Integer(3),-Integer(1)/Integer(3),-Integer(2)/Integer(3)]))
>>> g = H.fixed_point_set(); g
Geodesic in UHP from -1 to 1
>>> H(g.start()) == g.start()
True
>>> H(g.end()) == g.end()
True
>>> A = UHP.get_isometry(matrix(Integer(2),[Integer(0),Integer(1),Integer(1),
↪Integer(0)]))
>>> A.preserves_orientation()
False
>>> A.fixed_point_set()
Geodesic in UHP from 1 to -1
```

```
sage: B = UHP.get_isometry(identity_matrix(2))
sage: B.fixed_point_set()
Traceback (most recent call last):
...
ValueError: the identity transformation fixes the entire hyperbolic plane
```

```
>>> from sage.all import *
>>> B = UHP.get_isometry(identity_matrix(Integer(2)))
>>> B.fixed_point_set()
Traceback (most recent call last):
...
ValueError: the identity transformation fixes the entire hyperbolic plane
```

preserves_orientation()

Return `True` if `self` is orientation-preserving and `False` otherwise.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: A = identity_matrix(2)
```

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```
sage: UHP.get_isometry(A).preserves_orientation()
True
sage: B = matrix(2,[0,1,1,0])
sage: UHP.get_isometry(B).preserves_orientation()
False
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> A = identity_matrix(Integer(2))
>>> UHP.get_isometry(A).preserves_orientation()
True
>>> B = matrix(Integer(2),[Integer(0),Integer(1),Integer(1),Integer(0)])
>>> UHP.get_isometry(B).preserves_orientation()
False
```

repelling_fixed_point()

Return the repelling fixed point.

Otherwise, this raises a `ValueError`.

OUTPUT: a hyperbolic point

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: A = matrix(2,[4,0,0,1/4])
sage: UHP.get_isometry(A).repelling_fixed_point()
Boundary point in UHP 0
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> A = matrix(Integer(2),[Integer(4),Integer(0),Integer(0),Integer(1)/
↪Integer(4)])
>>> UHP.get_isometry(A).repelling_fixed_point()
Boundary point in UHP 0
```

translation_length()

For hyperbolic elements, return the translation length; otherwise, raise a `ValueError`.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_isometry(matrix(2,[2,0,0,1/2])).translation_length()
2*arccosh(5/4)
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.get_isometry(matrix(Integer(2),[Integer(2),Integer(0),Integer(0),
↪Integer(1)/Integer(2)])).translation_length()
2*arccosh(5/4)
```

```
sage: H = UHP.isometry_from_fixed_points(-1,1)
sage: p = UHP.get_point(exp(i*7*pi/8))
```

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```
sage: Hp = H(p)
sage: bool((UHP.dist(p, Hp) - H.translation_length()) < 10**-9)
True
```

```
>>> from sage.all import *
>>> H = UHP.isometry_from_fixed_points(-Integer(1),Integer(1))
>>> p = UHP.get_point(exp(i*Integer(7)*pi/Integer(8)))
>>> Hp = H(p)
>>> bool((UHP.dist(p, Hp) - H.translation_length()) < Integer(10)**-
↪Integer(9))
True
```

sage.geometry.hyperbolic_space.hyperbolic_isometry.**moebius_transform**(*A*, *z*)

Given a matrix A in $GL(2, \mathbf{C})$ and a point z in the complex plane return the Möbius transformation action of A on z.

INPUT:

- A – a 2×2 invertible matrix over the complex numbers
- z – a complex number or infinity

OUTPUT: a complex number or infinity

EXAMPLES:

```
sage: from sage.geometry.hyperbolic_space.hyperbolic_model import moebius_
↪transform
sage: moebius_transform(matrix(2,[1,2,3,4]),2 + I)
-2/109*I + 43/109
sage: y = var('y')
sage: moebius_transform(matrix(2,[1,0,0,1]),x + I*y)
x + I*y
```

```
>>> from sage.all import *
>>> from sage.geometry.hyperbolic_space.hyperbolic_model import moebius_transform
>>> moebius_transform(matrix(Integer(2),[Integer(1),Integer(2),Integer(3),
↪Integer(4)]),Integer(2) + I)
-2/109*I + 43/109
>>> y = var('y')
>>> moebius_transform(matrix(Integer(2),[Integer(1),Integer(0),Integer(0),
↪Integer(1)]),x + I*y)
x + I*y
```

The matrix must be square and 2×2:

```
sage: moebius_transform(matrix([[3,1,2],[1,2,5]]),I)
Traceback (most recent call last):
...
TypeError: A must be an invertible 2x2 matrix over the complex numbers or a␣
↪symbolic ring

sage: moebius_transform(identity_matrix(3),I)
Traceback (most recent call last):
```

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```
...
TypeError: A must be an invertible 2x2 matrix over the complex numbers or a␣
↪symbolic ring
```

```
>>> from sage.all import *
>>> moebius_transform(matrix([[Integer(3),Integer(1),Integer(2)],[Integer(1),
↪Integer(2),Integer(5)]]),I)
Traceback (most recent call last):
...
TypeError: A must be an invertible 2x2 matrix over the complex numbers or a␣
↪symbolic ring

>>> moebius_transform(identity_matrix(Integer(3)),I)
Traceback (most recent call last):
...
TypeError: A must be an invertible 2x2 matrix over the complex numbers or a␣
↪symbolic ring
```

The matrix can be symbolic or can be a matrix over the real or complex numbers, but must be provably invertible:

```
sage: a,b,c,d = var('a,b,c,d')
sage: moebius_transform(matrix(2,[a,b,c,d]),I)
(I*a + b)/(I*c + d)
sage: moebius_transform(matrix(2,[1,b,c,b*c+1]),I)
(b + I)/(b*c + I*c + 1)
sage: moebius_transform(matrix(2,[0,0,0,0]),I)
Traceback (most recent call last):
...
TypeError: A must be an invertible 2x2 matrix over the complex numbers or a␣
↪symbolic ring
```

```
>>> from sage.all import *
>>> a,b,c,d = var('a,b,c,d')
>>> moebius_transform(matrix(Integer(2),[a,b,c,d]),I)
(I*a + b)/(I*c + d)
>>> moebius_transform(matrix(Integer(2),[Integer(1),b,c,b*c+Integer(1)]),I)
(b + I)/(b*c + I*c + 1)
>>> moebius_transform(matrix(Integer(2),[Integer(0),Integer(0),Integer(0),
↪Integer(0)]),I)
Traceback (most recent call last):
...
TypeError: A must be an invertible 2x2 matrix over the complex numbers or a␣
↪symbolic ring
```

CHAPTER

THREE

HYPERBOLIC GEODESICS

This module implements the abstract base class for geodesics in hyperbolic space of arbitrary dimension. It also contains the implementations for specific models of hyperbolic geometry.

AUTHORS:

- Greg Laun (2013): initial version

EXAMPLES:

We can construct geodesics in the upper half plane model, abbreviated UHP for convenience:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(2, 3)
sage: g
Geodesic in UHP from 2 to 3
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(Integer(2), Integer(3))
>>> g
Geodesic in UHP from 2 to 3
```

This geodesic can be plotted using `plot()`, in this example we will show the axis.

```
sage: g.plot(axes=True)                                                    #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

```
>>> from sage.all import *
>>> g.plot(axes=True)                                                      #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

```
sage: g = HyperbolicPlane().UHP().get_geodesic(I, 3 + I)
sage: g.length()
arccosh(11/2)
sage: g.plot(axes=True)                                                    #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(I, Integer(3) + I)
>>> g.length()
```

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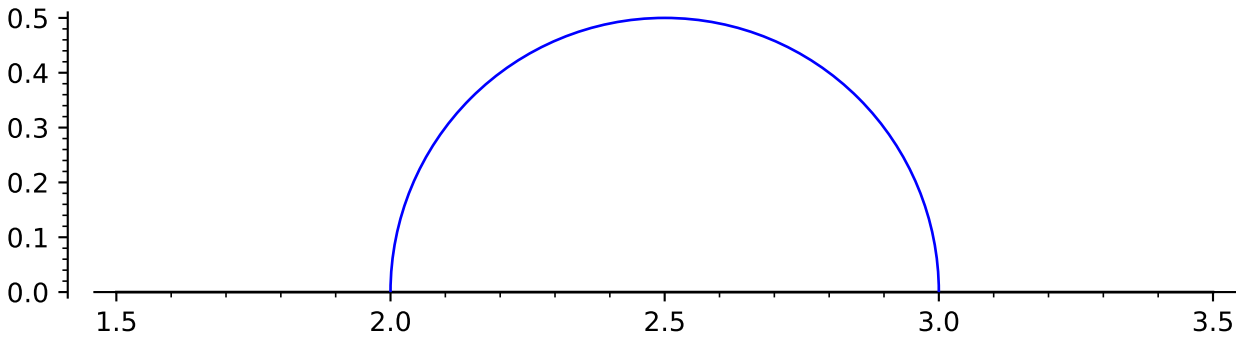
0.5
0.4
0.3
0.2
0.1
0.0
1.5
2.0
2.5
3.0
3.5

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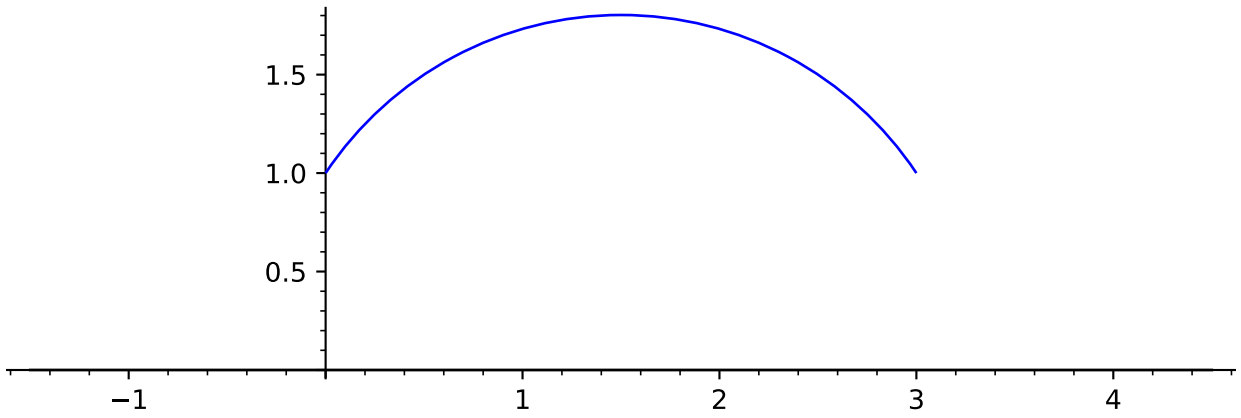
```
arccosh(11/2)
>>> g.plot(axes=True)                                                    #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

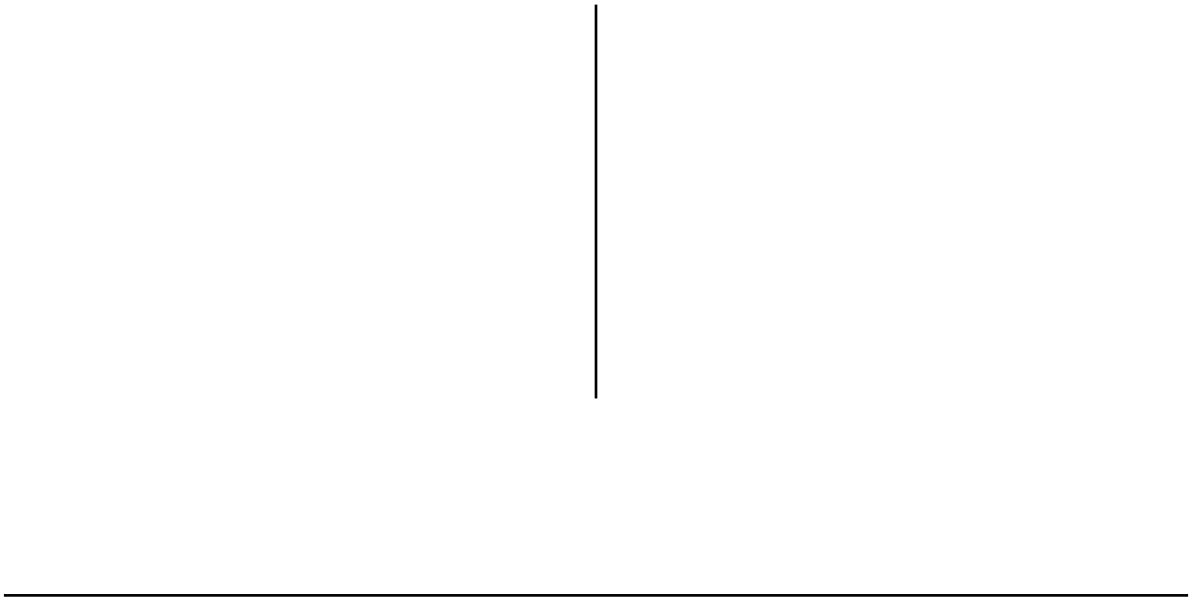


Geodesics of both types in UHP are supported:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(I, 3*I)
sage: g
Geodesic in UHP from I to 3*I
sage: g.plot()                                                           #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(I, Integer(3)*I)
>>> g
Geodesic in UHP from I to 3*I
>>> g.plot()                                                             #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

Geodesics are oriented, which means that two geodesics with the same graph will only be equal if their starting and ending



points are the same:

```
sage: g1 = HyperbolicPlane().UHP().get_geodesic(1,2)
sage: g2 = HyperbolicPlane().UHP().get_geodesic(2,1)
sage: g1 == g2
False
```

```
>>> from sage.all import *
>>> g1 = HyperbolicPlane().UHP().get_geodesic(Integer(1),Integer(2))
>>> g2 = HyperbolicPlane().UHP().get_geodesic(Integer(2),Integer(1))
>>> g1 == g2
False
```

Todo

Implement a parent for all geodesics of the hyperbolic plane? Or implement geodesics as a parent in the subobjects category?

class sage.geometry.hyperbolic_space.hyperbolic_geodesic.**HyperbolicGeodesic**(*model*, *start*, *end*, ***graphics_options*)

Bases: SageObject

Abstract base class for oriented geodesics that are not necessarily complete.

INPUT:

- `start` – a HyperbolicPoint or coordinates of a point in hyperbolic space representing the start of the geodesic
- `end` – a HyperbolicPoint or coordinates of a point in hyperbolic space representing the end of the geodesic

EXAMPLES:

We can construct a hyperbolic geodesic in any model:

```
sage: HyperbolicPlane().UHP().get_geodesic(1, 0)
Geodesic in UHP from 1 to 0
sage: HyperbolicPlane().PD().get_geodesic(1, 0)
Geodesic in PD from 1 to 0
sage: HyperbolicPlane().KM().get_geodesic((0,1/2), (1/2, 0))
Geodesic in KM from (0, 1/2) to (1/2, 0)
sage: HyperbolicPlane().HM().get_geodesic((0,0,1), (0,1, sqrt(2)))
Geodesic in HM from (0, 0, 1) to (0, 1, sqrt(2))
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_geodesic(Integer(1), Integer(0))
Geodesic in UHP from 1 to 0
>>> HyperbolicPlane().PD().get_geodesic(Integer(1), Integer(0))
Geodesic in PD from 1 to 0
>>> HyperbolicPlane().KM().get_geodesic((Integer(0),Integer(1)/Integer(2)),␣
↪(Integer(1)/Integer(2), Integer(0)))
Geodesic in KM from (0, 1/2) to (1/2, 0)
>>> HyperbolicPlane().HM().get_geodesic((Integer(0),Integer(0),Integer(1)),␣
↪(Integer(0),Integer(1), sqrt(Integer(2))))
Geodesic in HM from (0, 0, 1) to (0, 1, sqrt(2))
```

angle(*other*)

Return the angle between any two given geodesics if they intersect.

INPUT:

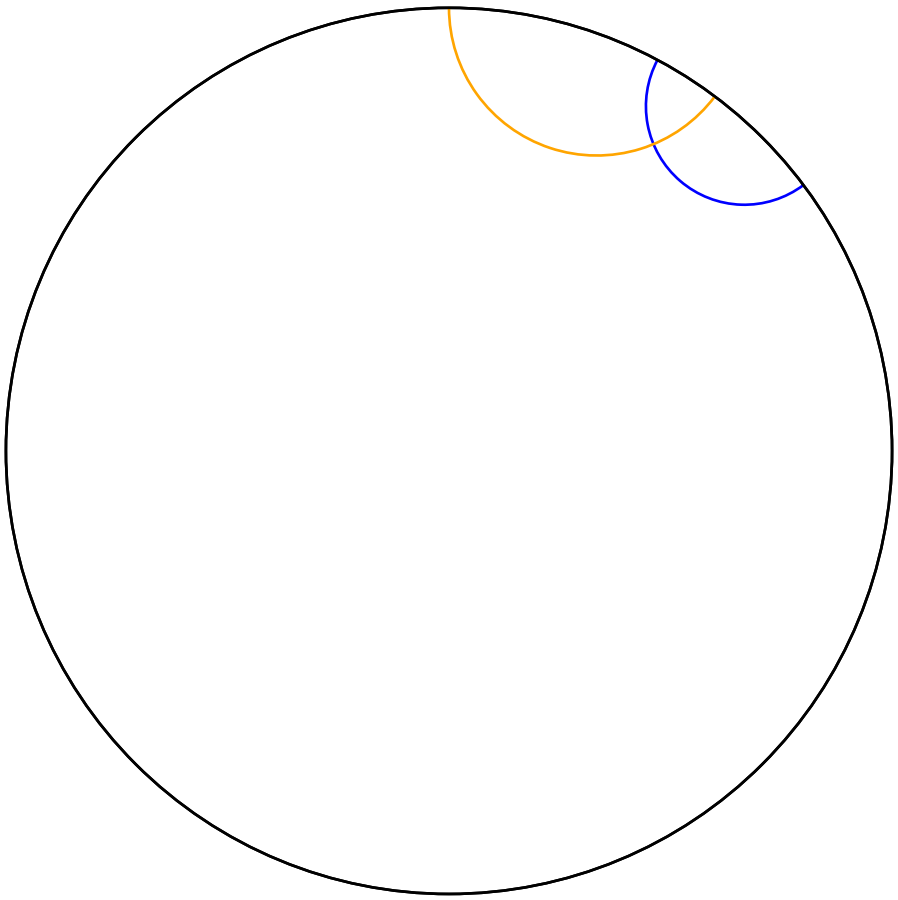
- `other` – a hyperbolic geodesic in the same model as `self`

OUTPUT: the angle in radians between the two given geodesics

EXAMPLES:

```
sage: PD = HyperbolicPlane().PD()
sage: g = PD.get_geodesic(3/5*I + 4/5, 15/17*I + 8/17)
sage: h = PD.get_geodesic(4/5*I + 3/5, I)
sage: g.angle(h)
1/2*pi
```

```
>>> from sage.all import *
>>> PD = HyperbolicPlane().PD()
>>> g = PD.get_geodesic(Integer(3)/Integer(5)*I + Integer(4)/Integer(5),␣
↪Integer(15)/Integer(17)*I + Integer(8)/Integer(17))
>>> h = PD.get_geodesic(Integer(4)/Integer(5)*I + Integer(3)/Integer(5), I)
>>> g.angle(h)
1/2*pi
```



common_perpendicula(*other*)

Return the unique hyperbolic geodesic perpendicular to two given geodesics, if such a geodesic exists. If none exists, raise a `ValueError`.

INPUT:

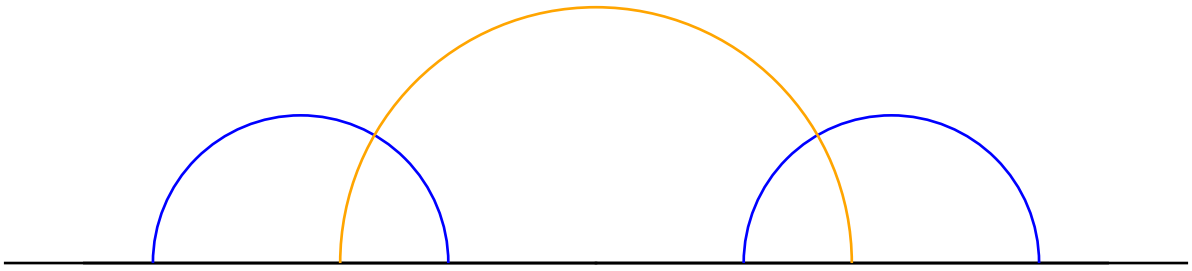
- `other` – a hyperbolic geodesic in the same model as `self`

OUTPUT: a hyperbolic geodesic

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(2,3)
sage: h = HyperbolicPlane().UHP().get_geodesic(4,5)
sage: g.common_perpendicular(h)
Geodesic in UHP from 1/2*sqrt(3) + 7/2 to -1/2*sqrt(3) + 7/2
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(Integer(2),Integer(3))
>>> h = HyperbolicPlane().UHP().get_geodesic(Integer(4),Integer(5))
>>> g.common_perpendicular(h)
Geodesic in UHP from 1/2*sqrt(3) + 7/2 to -1/2*sqrt(3) + 7/2
```



It is an error to ask for the common perpendicular of two intersecting geodesics:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(2,4)
sage: h = HyperbolicPlane().UHP().get_geodesic(3, infinity)
sage: g.common_perpendicular(h)
Traceback (most recent call last):
...
ValueError: geodesics intersect; no common perpendicular exists
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(Integer(2),Integer(4))
>>> h = HyperbolicPlane().UHP().get_geodesic(Integer(3), infinity)
>>> g.common_perpendicular(h)
Traceback (most recent call last):
...
ValueError: geodesics intersect; no common perpendicular exists
```

complete()

Return the geodesic with ideal endpoints in bounded models. Raise a `NotImplementedError` in models that are not bounded. In the following examples we represent complete geodesics by a dashed line.

EXAMPLES:

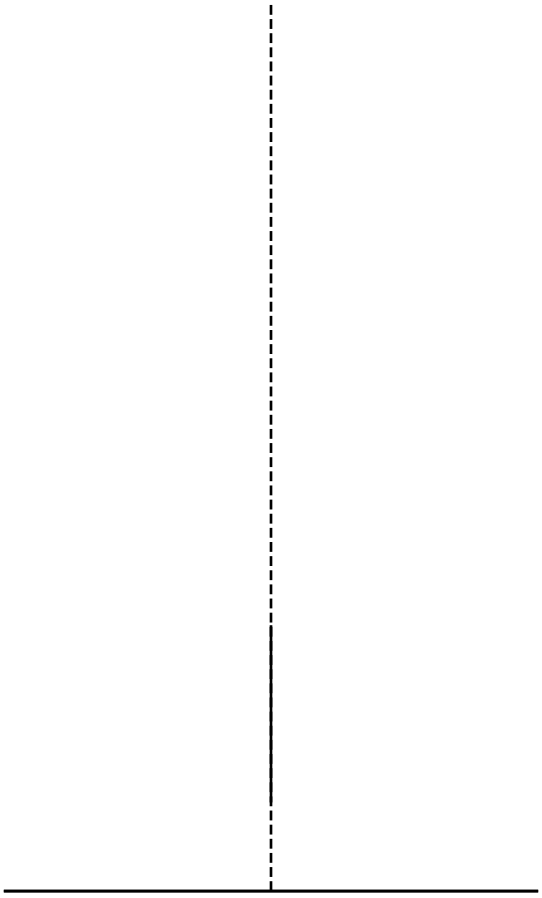
```
sage: H = HyperbolicPlane()
sage: UHP = H.UHP()
sage: UHP.get_geodesic(1 + I, 1 + 3*I).complete()
Geodesic in UHP from 1 to +Infinity
```

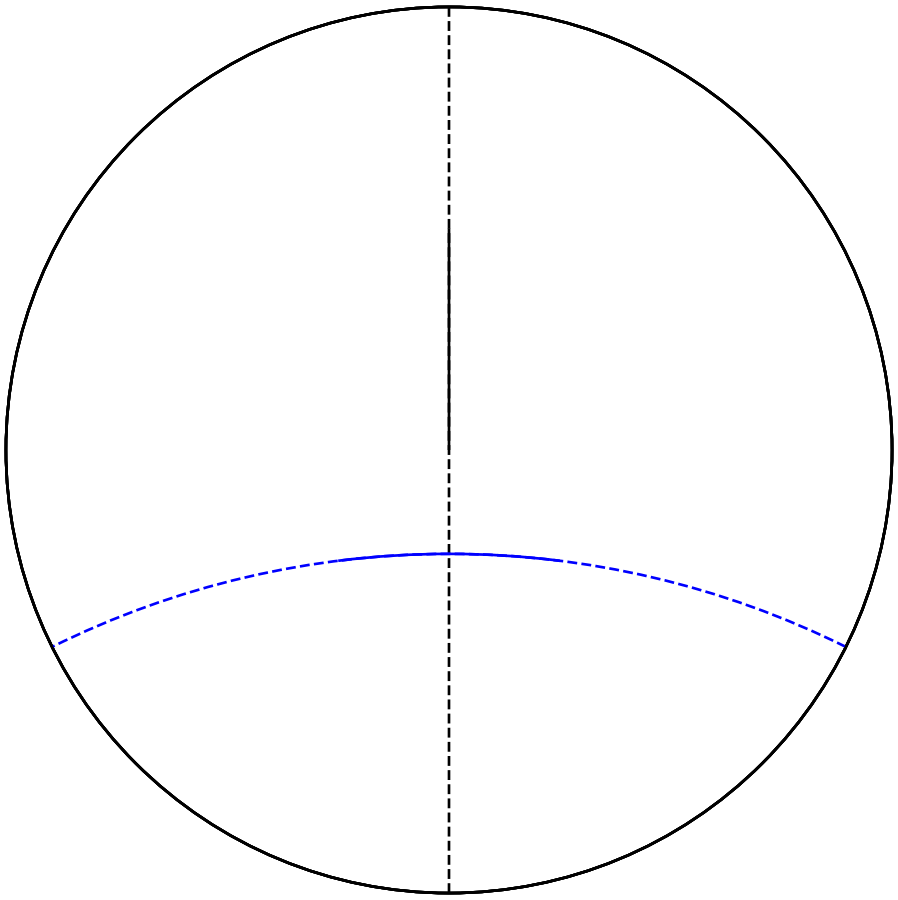
```
>>> from sage.all import *
>>> H = HyperbolicPlane()
>>> UHP = H.UHP()
>>> UHP.get_geodesic(Integer(1) + I, Integer(1) + Integer(3)*I).complete()
Geodesic in UHP from 1 to +Infinity
```

```
sage: PD = H.PD()
sage: PD.get_geodesic(0, I/2).complete()
Geodesic in PD from -I to I
sage: PD.get_geodesic(0.25*(-1-I),0.25*(1-I)).complete()
Geodesic in PD from -0.895806416477617 - 0.444444444444444*I to 0.
↪895806416477617 - 0.444444444444444*I
```

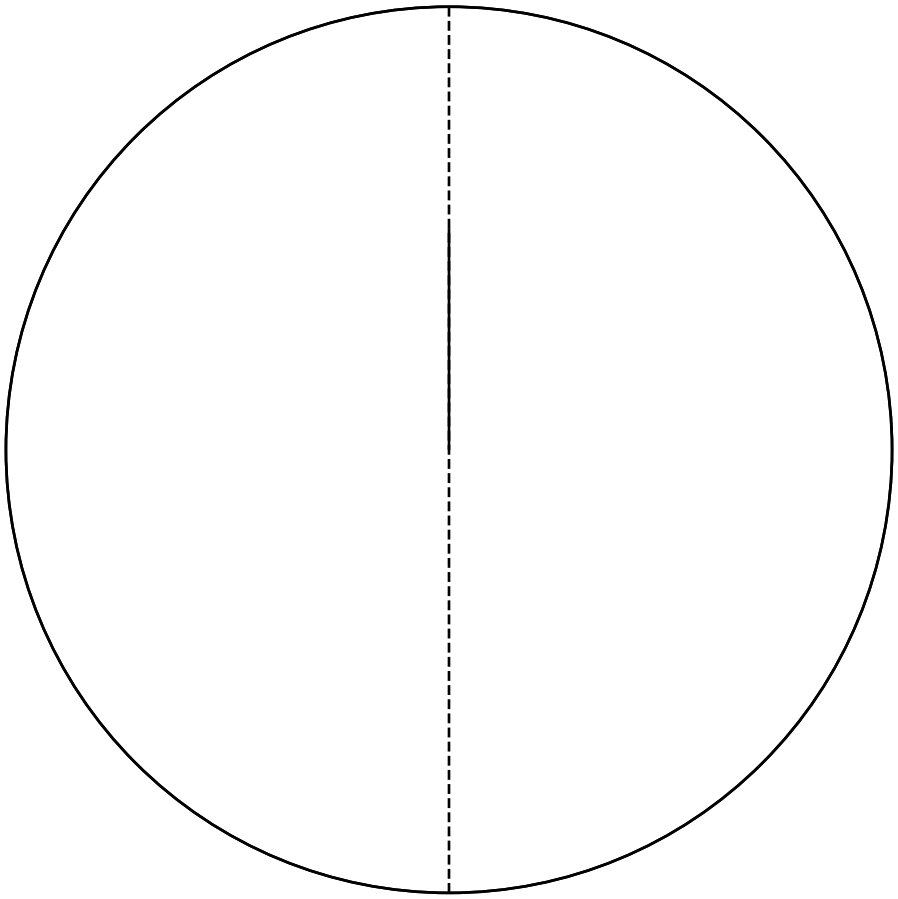
```
>>> from sage.all import *
>>> PD = H.PD()
>>> PD.get_geodesic(Integer(0), I/Integer(2)).complete()
Geodesic in PD from -I to I
>>> PD.get_geodesic(RealNumber('0.25')*(-Integer(1)-I),RealNumber('0.25
↪')*(Integer(1)-I)).complete()
Geodesic in PD from -0.895806416477617 - 0.444444444444444*I to 0.
↪895806416477617 - 0.444444444444444*I
```

```
sage: KM = H.KM()
sage: KM.get_geodesic((0,0), (0,1/2)).complete()
Geodesic in KM from (0, -1) to (0, 1)
```





```
>>> from sage.all import *
>>> KM = H.KM()
>>> KM.get_geodesic((Integer(0),Integer(0)), (Integer(0),Integer(1)/
↪Integer(2))).complete()
Geodesic in KM from (0, -1) to (0, 1)
```

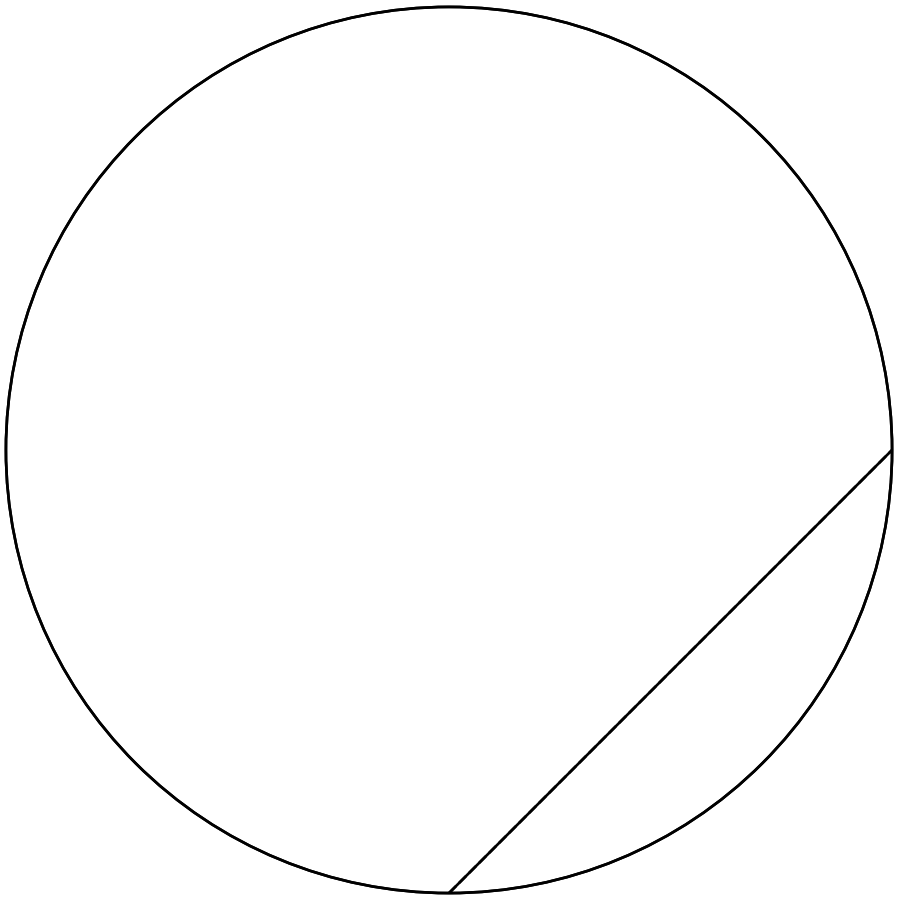


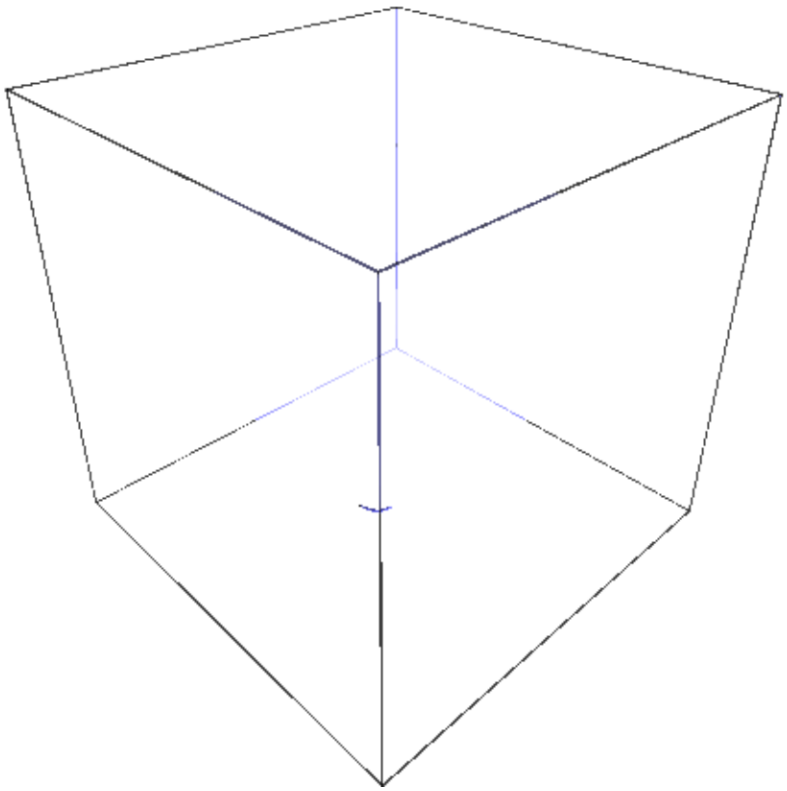
```
sage: KM.get_geodesic(-I, 1).complete()
Geodesic in KM from -I to 1
```

```
>>> from sage.all import *
>>> KM.get_geodesic(-I, Integer(1)).complete()
Geodesic in KM from -I to 1
```

```
sage: HM = H.HM()
sage: HM.get_geodesic((0,0,1), (1, 0, sqrt(2))).complete()
Geodesic in HM from (0, 0, 1) to (1, 0, sqrt(2))
```

```
>>> from sage.all import *
>>> HM = H.HM()
>>> HM.get_geodesic((Integer(0),Integer(0),Integer(1)), (Integer(1),␣
↪Integer(0), sqrt(Integer(2)))).complete()
Geodesic in HM from (0, 0, 1) to (1, 0, sqrt(2))
```





```
sage: g = HM.get_geodesic((0,0,1), (1, 0, sqrt(2))).complete()
sage: g.is_complete()
True
```

```
>>> from sage.all import *
>>> g = HM.get_geodesic((Integer(0),Integer(0),Integer(1)), (Integer(1),␣
↪Integer(0), sqrt(Integer(2)))).complete()
>>> g.is_complete()
True
```

dist(*other*)

Return the hyperbolic distance from a given hyperbolic geodesic to another geodesic or point.

INPUT:

- `other` – a hyperbolic geodesic or hyperbolic point in the same model

OUTPUT: the hyperbolic distance

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(2, 4.0)
sage: h = HyperbolicPlane().UHP().get_geodesic(5, 7.0)
sage: bool(abs(g.dist(h).n() - 1.92484730023841) < 10**-9)
True
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(Integer(2), RealNumber('4.0'))
>>> h = HyperbolicPlane().UHP().get_geodesic(Integer(5), RealNumber('7.0'))
>>> bool(abs(g.dist(h).n() - RealNumber('1.92484730023841')) < Integer(10)**-
↪Integer(9))
True
```

If the second object is a geodesic ultraparallel to the first, or if it is a point on the boundary that is not one of the first object's endpoints, then return +infinity

```
sage: g = HyperbolicPlane().UHP().get_geodesic(2, 2+I)
sage: p = HyperbolicPlane().UHP().get_point(5)
sage: g.dist(p)
+Infinity
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(Integer(2), Integer(2)+I)
>>> p = HyperbolicPlane().UHP().get_point(Integer(5))
>>> g.dist(p)
+Infinity
```

end()

Return the starting point of the geodesic.

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(I, 3*I)
sage: g.end()
Point in UHP 3*I
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(I, Integer(3)*I)
>>> g.end()
Point in UHP 3*I
```

endpoints()

Return a list containing the start and endpoints.

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(I, 3*I)
sage: g.endpoints()
[Point in UHP I, Point in UHP 3*I]
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(I, Integer(3)*I)
>>> g.endpoints()
[Point in UHP I, Point in UHP 3*I]
```

graphics_options()

Return the graphics options of `self`.

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(I, 2*I, color='red')
sage: g.graphics_options()
{'color': 'red'}
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(I, Integer(2)*I, color='red')
>>> g.graphics_options()
{'color': 'red'}
```

ideal_endpoints()

Return the ideal endpoints in bounded models. Raise a `NotImplementedError` in models that are not bounded.

EXAMPLES:

```
sage: H = HyperbolicPlane()
sage: UHP = H.UHP()
sage: UHP.get_geodesic(1 + I, 1 + 3*I).ideal_endpoints()
[Boundary point in UHP 1, Boundary point in UHP +Infinity]

sage: PD = H.PD()
sage: PD.get_geodesic(0, I/2).ideal_endpoints()
[Boundary point in PD -I, Boundary point in PD I]

sage: KM = H.KM()
sage: KM.get_geodesic((0,0), (0, 1/2)).ideal_endpoints()
[Boundary point in KM (0, -1), Boundary point in KM (0, 1)]

sage: HM = H.HM()
```

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```
sage: HM.get_geodesic((0,0,1), (1, 0, sqrt(2))).ideal_endpoints()
Traceback (most recent call last):
...
NotImplementedError: boundary points are not implemented in
 the HM model
```

```
>>> from sage.all import *
>>> H = HyperbolicPlane()
>>> UHP = H.UHP()
>>> UHP.get_geodesic(Integer(1) + I, Integer(1) + Integer(3)*I).ideal_
↪endpoints()
[Boundary point in UHP 1, Boundary point in UHP +Infinity]

>>> PD = H.PD()
>>> PD.get_geodesic(Integer(0), I/Integer(2)).ideal_endpoints()
[Boundary point in PD -I, Boundary point in PD I]

>>> KM = H.KM()
>>> KM.get_geodesic((Integer(0),Integer(0)), (Integer(0), Integer(1)/
↪Integer(2))).ideal_endpoints()
[Boundary point in KM (0, -1), Boundary point in KM (0, 1)]

>>> HM = H.HM()
>>> HM.get_geodesic((Integer(0),Integer(0),Integer(1)), (Integer(1),␣
↪Integer(0), sqrt(Integer(2)))).ideal_endpoints()
Traceback (most recent call last):
...
NotImplementedError: boundary points are not implemented in
 the HM model
```

intersection(*other*)

Return the point of intersection of two geodesics (if such a point exists).

INPUT:

- `other` – a hyperbolic geodesic in the same model as `self`

OUTPUT: a hyperbolic point or geodesic

EXAMPLES:

```
sage: PD = HyperbolicPlane().PD()
```

```
>>> from sage.all import *
>>> PD = HyperbolicPlane().PD()
```

is_asymptotically_parallel(*other*)

Return `True` if `self` and `other` are asymptotically parallel and `False` otherwise.

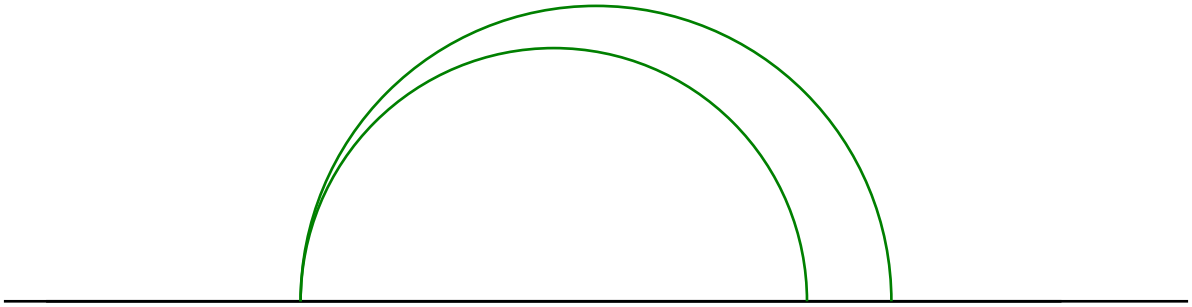
INPUT:

- `other` – a hyperbolic geodesic

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,5)
sage: h = HyperbolicPlane().UHP().get_geodesic(-2,4)
sage: g.is_asymptotically_parallel(h)
True
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(-Integer(2),Integer(5))
>>> h = HyperbolicPlane().UHP().get_geodesic(-Integer(2),Integer(4))
>>> g.is_asymptotically_parallel(h)
True
```



Ultraparallel geodesics are not asymptotically parallel:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,5)
sage: h = HyperbolicPlane().UHP().get_geodesic(-1,4)
sage: g.is_asymptotically_parallel(h)
False
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(-Integer(2),Integer(5))
>>> h = HyperbolicPlane().UHP().get_geodesic(-Integer(1),Integer(4))
>>> g.is_asymptotically_parallel(h)
```

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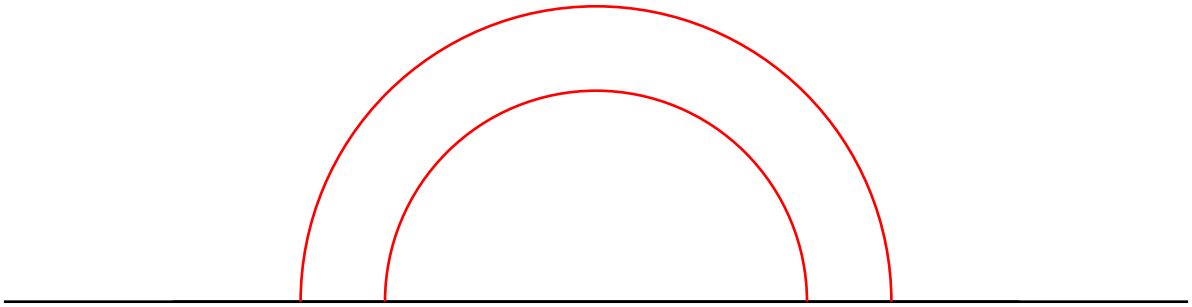
```
False
```



No hyperbolic geodesic is asymptotically parallel to itself:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,5)
sage: g.is_asymptotically_parallel(g)
False
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(-Integer(2),Integer(5))
>>> g.is_asymptotically_parallel(g)
False
```

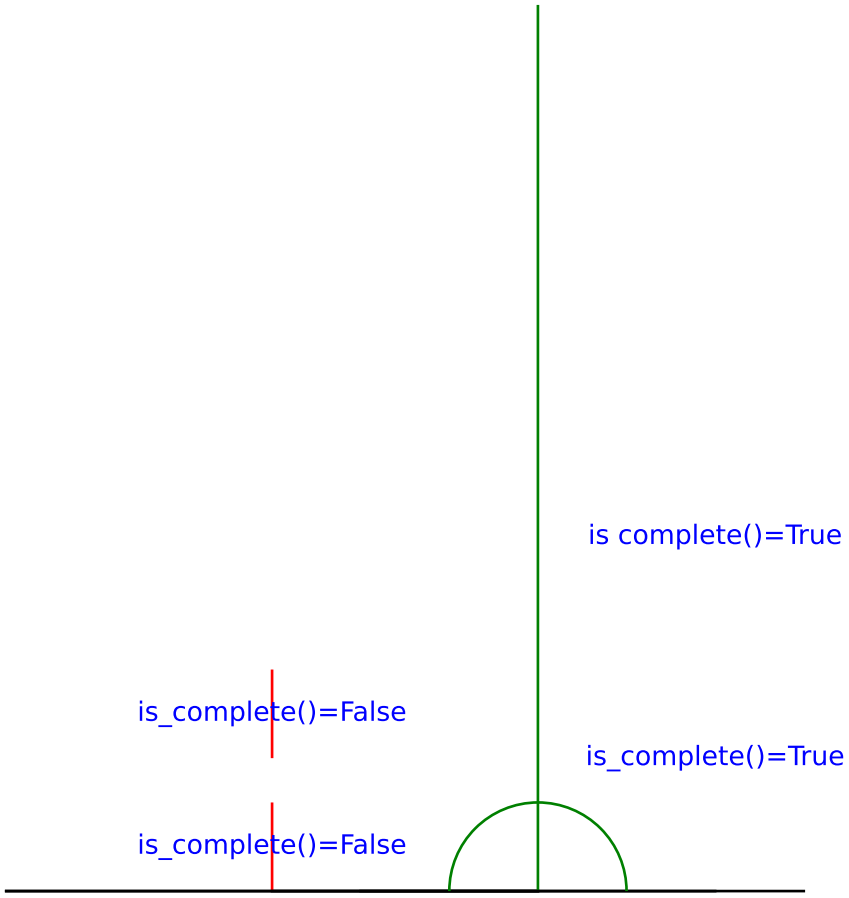
is_complete()

Return `True` if `self` is a complete geodesic (that is, both endpoints are on the ideal boundary) and `False` otherwise.

If we represent complete geodesics using green color and incomplete using red colors we have the following graphic:

Notice, that there is no visual indication that the *vertical* geodesic is complete

EXAMPLES:


is complete()=True
is_complete()=False
is_complete()=True
is_complete()=False

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_geodesic(1.5*I, 2.5*I).is_complete()
False
sage: UHP.get_geodesic(0, I).is_complete()
False
sage: UHP.get_geodesic(3, infinity).is_complete()
True
sage: UHP.get_geodesic(2,5).is_complete()
True
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.get_geodesic(RealNumber('1.5')*I, RealNumber('2.5')*I).is_complete()
False
>>> UHP.get_geodesic(Integer(0), I).is_complete()
False
>>> UHP.get_geodesic(Integer(3), infinity).is_complete()
True
>>> UHP.get_geodesic(Integer(2),Integer(5)).is_complete()
True
```

is_parallel(*other*)

Return `True` if the two given hyperbolic geodesics are either ultra parallel or asymptotically parallel and `False` otherwise.

INPUT:

- `other` – a hyperbolic geodesic in any model

OUTPUT:

`True` if the given geodesics are either ultra parallel or asymptotically parallel, `False` if not.

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,5)
sage: h = HyperbolicPlane().UHP().get_geodesic(5,12)
sage: g.is_parallel(h)
True
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(-Integer(2),Integer(5))
>>> h = HyperbolicPlane().UHP().get_geodesic(Integer(5),Integer(12))
>>> g.is_parallel(h)
True
```

```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,5)
sage: h = HyperbolicPlane().UHP().get_geodesic(-2,4)
sage: g.is_parallel(h)
True
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(-Integer(2),Integer(5))
>>> h = HyperbolicPlane().UHP().get_geodesic(-Integer(2),Integer(4))
```

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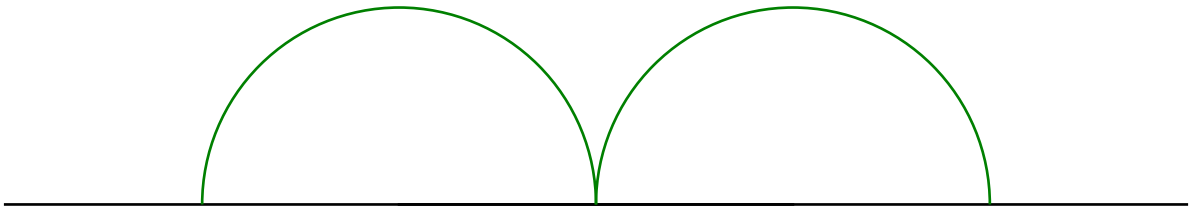

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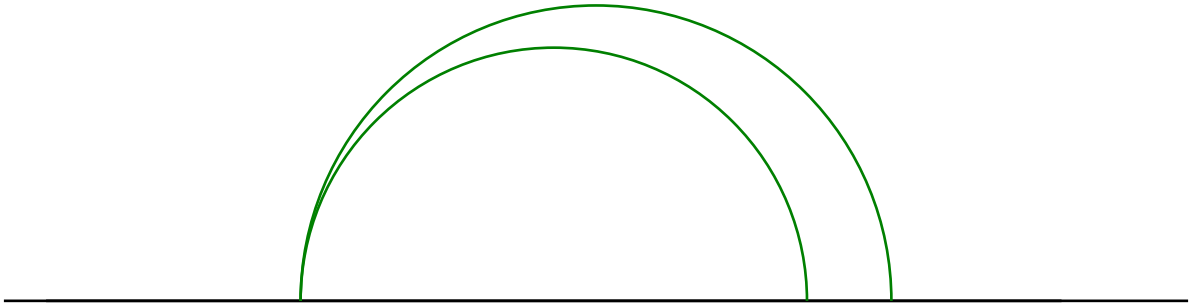
```
>>> g.is_parallel(h)
True
```



```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,2)
sage: h = HyperbolicPlane().UHP().get_geodesic(-1,4)
sage: g.is_parallel(h)
False
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(-Integer(2),Integer(2))
>>> h = HyperbolicPlane().UHP().get_geodesic(-Integer(1),Integer(4))
>>> g.is_parallel(h)
False
```

No hyperbolic geodesic is either ultra parallel or asymptotically parallel to itself:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,5)
sage: g.is_parallel(g)
False
```

```
>>> from sage.all import *
```

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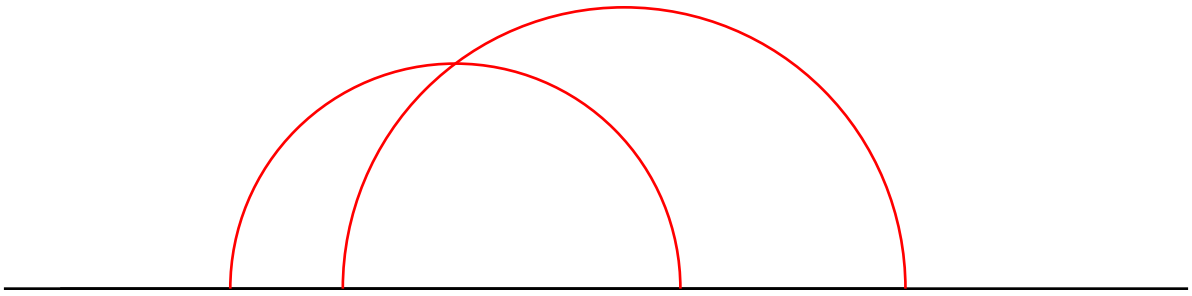

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```
>>> g = HyperbolicPlane().UHP().get_geodesic(-Integer(2),Integer(5))
>>> g.is_parallel(g)
False
```

is_ultra_parallel(*other*)

Return `True` if `self` and `other` are ultra parallel and `False` otherwise.

INPUT:

- `other` – a hyperbolic geodesic

EXAMPLES:

```
sage: from sage.geometry.hyperbolic_space.hyperbolic_geodesic \
....:   import *
sage: g = HyperbolicPlane().UHP().get_geodesic(0,1)
sage: h = HyperbolicPlane().UHP().get_geodesic(-3,-1)
sage: g.is_ultra_parallel(h)
True
```

```
>>> from sage.all import *
>>> from sage.geometry.hyperbolic_space.hyperbolic_geodesic   import *
>>> g = HyperbolicPlane().UHP().get_geodesic(Integer(0),Integer(1))
>>> h = HyperbolicPlane().UHP().get_geodesic(-Integer(3),-Integer(1))
>>> g.is_ultra_parallel(h)
True
```

```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,5)
sage: h = HyperbolicPlane().UHP().get_geodesic(2,6)
sage: g.is_ultra_parallel(h)
False
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(-Integer(2),Integer(5))
>>> h = HyperbolicPlane().UHP().get_geodesic(Integer(2),Integer(6))
>>> g.is_ultra_parallel(h)
False
```

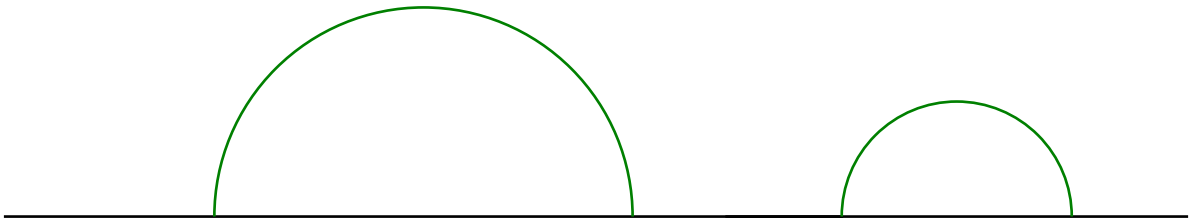
```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,5)
sage: g.is_ultra_parallel(g)
False
```

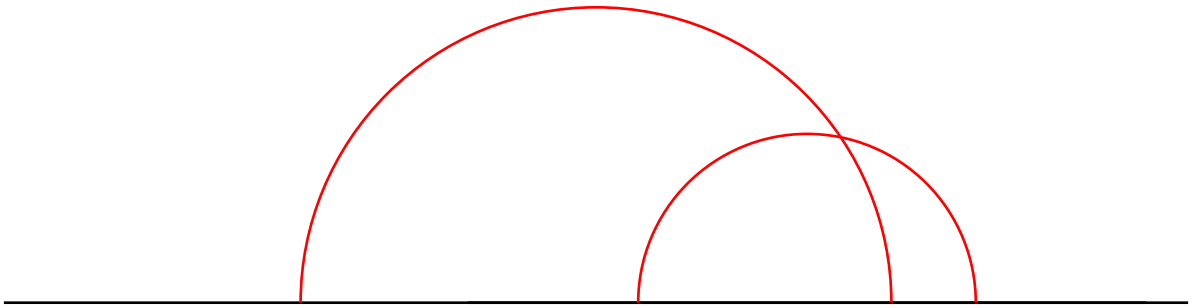
```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(-Integer(2),Integer(5))
>>> g.is_ultra_parallel(g)
False
```

length()

Return the Hyperbolic length of the hyperbolic line segment.

EXAMPLES:





```
sage: g = HyperbolicPlane().UHP().get_geodesic(2 + I, 3 + I/2)
sage: g.length()
arccosh(9/4)
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(Integer(2) + I, Integer(3) + I/
↪Integer(2))
>>> g.length()
arccosh(9/4)
```

midpoint()

Return the (hyperbolic) midpoint of a hyperbolic line segment.

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().random_geodesic()
sage: m = g.midpoint()
sage: end1, end2 = g.endpoints()
sage: bool(abs(m.dist(end1) - m.dist(end2)) < 10**-9)
True
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().random_geodesic()
>>> m = g.midpoint()
>>> end1, end2 = g.endpoints()
>>> bool(abs(m.dist(end1) - m.dist(end2)) < Integer(10)**-Integer(9))
True
```

Complete geodesics have no midpoint:

```
sage: HyperbolicPlane().UHP().get_geodesic(0,2).midpoint()
Traceback (most recent call last):
...
ValueError: the length must be finite
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_geodesic(Integer(0),Integer(2)).midpoint()
Traceback (most recent call last):
...
ValueError: the length must be finite
```

model()

Return the model to which the *HyperbolicGeodesic* belongs.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_geodesic(I, 2*I).model()
Hyperbolic plane in the Upper Half Plane Model

sage: PD = HyperbolicPlane().PD()
sage: PD.get_geodesic(0, I/2).model()
Hyperbolic plane in the Poincare Disk Model
```

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```
sage: KM = HyperbolicPlane().KM()
sage: KM.get_geodesic((0, 0), (0, 1/2)).model()
Hyperbolic plane in the Klein Disk Model

sage: HM = HyperbolicPlane().HM()
sage: HM.get_geodesic((0, 0, 1), (0, 1, sqrt(2))).model()
Hyperbolic plane in the Hyperboloid Model
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.get_geodesic(I, Integer(2)*I).model()
Hyperbolic plane in the Upper Half Plane Model

>>> PD = HyperbolicPlane().PD()
>>> PD.get_geodesic(Integer(0), I/Integer(2)).model()
Hyperbolic plane in the Poincare Disk Model

>>> KM = HyperbolicPlane().KM()
>>> KM.get_geodesic((Integer(0), Integer(0)), (Integer(0), Integer(1)/
↪Integer(2))).model()
Hyperbolic plane in the Klein Disk Model

>>> HM = HyperbolicPlane().HM()
>>> HM.get_geodesic((Integer(0), Integer(0), Integer(1)), (Integer(0),␣
↪Integer(1), sqrt(Integer(2)))).model()
Hyperbolic plane in the Hyperboloid Model
```

perpendicular_bisector()

Return the perpendicular bisector of `self` if `self` has finite length. Here distance is hyperbolic distance.

EXAMPLES:

```
sage: PD = HyperbolicPlane().PD()
sage: g = PD.get_geodesic(-0.3+0.4*I,+0.7-0.1*I)
sage: h = g.perpendicular_bisector().complete()
sage: P = g.plot(color='blue')+h.plot(color='orange');P
Graphics object consisting of 4 graphics primitives
```

```
>>> from sage.all import *
>>> PD = HyperbolicPlane().PD()
>>> g = PD.get_geodesic(-RealNumber('0.3')+RealNumber('0.4')*I,+RealNumber('0.
↪7')-RealNumber('0.1')*I)
>>> h = g.perpendicular_bisector().complete()
>>> P = g.plot(color='blue')+h.plot(color='orange');P
Graphics object consisting of 4 graphics primitives
```

Complete geodesics cannot be bisected:

```
sage: g = HyperbolicPlane().PD().get_geodesic(0, 1)
sage: g.perpendicular_bisector()
Traceback (most recent call last):
```

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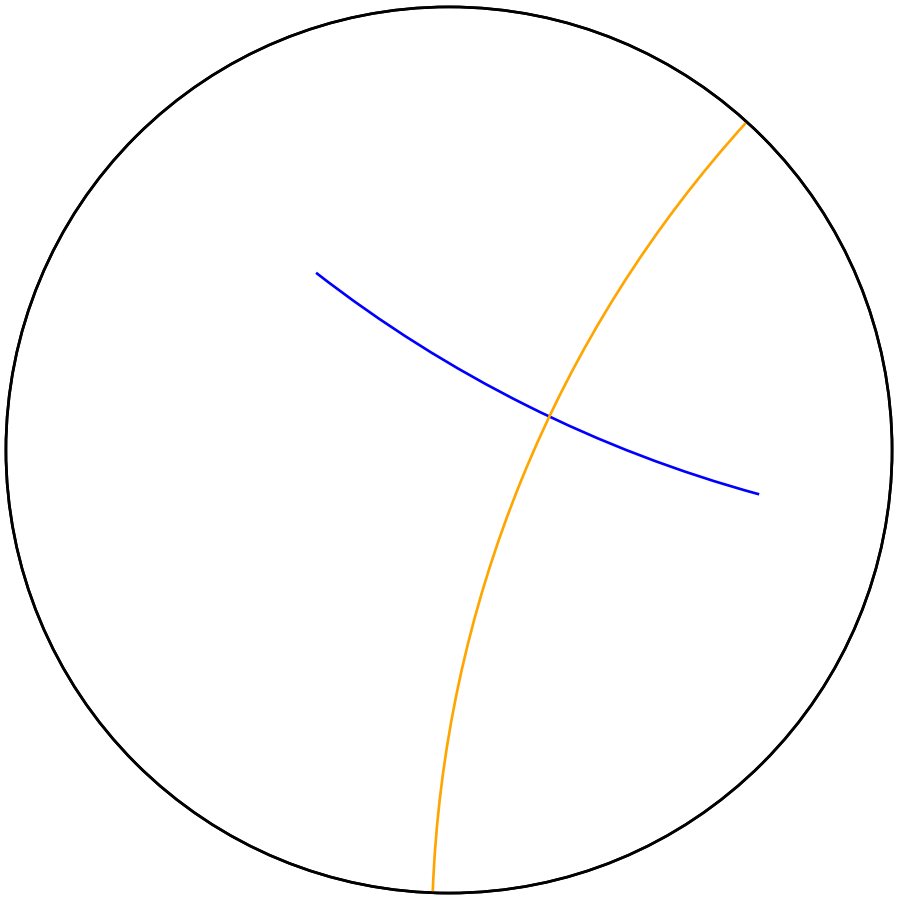

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```
...
ValueError: the length must be finite
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().PD().get_geodesic(Integer(0), Integer(1))
>>> g.perpendicular_bisector()
Traceback (most recent call last):
...
ValueError: the length must be finite
```

reflection_involution()

Return the involution fixing `self`.

EXAMPLES:

```
sage: H = HyperbolicPlane()
sage: gU = H.UHP().get_geodesic(2,4)
sage: RU = gU.reflection_involution(); RU
Isometry in UHP
[ 3 -8]
[ 1 -3]

sage: RU*gU == gU
True

sage: gP = H.PD().get_geodesic(0, I)
sage: RP = gP.reflection_involution(); RP
Isometry in PD
[ 1  0]
[ 0 -1]

sage: RP*gP == gP
True

sage: gK = H.KM().get_geodesic((0,0), (0,1))
sage: RK = gK.reflection_involution(); RK
Isometry in KM
[-1  0  0]
[ 0  1  0]
[ 0  0  1]

sage: RK*gK == gK
True

sage: HM = H.HM()
sage: g = HM.get_geodesic((0,0,1), (1,0, n(sqrt(2))))
sage: A = g.reflection_involution()
sage: B = diagonal_matrix([1, -1, 1])
sage: bool((B - A.matrix()).norm() < 10**-9)                                    #␣
↪needs scipy
True
```

```
>>> from sage.all import *
>>> H = HyperbolicPlane()
>>> gU = H.UHP().get_geodesic(Integer(2),Integer(4))
>>> RU = gU.reflection_involution(); RU
Isometry in UHP
[ 3 -8]
[ 1 -3]

>>> RU*gU == gU
True

>>> gP = H.PD().get_geodesic(Integer(0), I)
>>> RP = gP.reflection_involution(); RP
Isometry in PD
[ 1  0]
[ 0 -1]

>>> RP*gP == gP
True

>>> gK = H.KM().get_geodesic((Integer(0),Integer(0)), (Integer(0),Integer(1)))
>>> RK = gK.reflection_involution(); RK
Isometry in KM
[-1  0  0]
[ 0  1  0]
[ 0  0  1]

>>> RK*gK == gK
True

>>> HM = H.HM()
>>> g = HM.get_geodesic((Integer(0),Integer(0),Integer(1)), (Integer(1),
↪Integer(0), n(sqrt(Integer(2)))))
>>> A = g.reflection_involution()
>>> B = diagonal_matrix([Integer(1), -Integer(1), Integer(1)])
>>> bool((B - A.matrix()).norm() < Integer(10)**-Integer(9))                ␣
↪              # needs scipy
True
```

The above tests go through the Upper Half Plane. It remains to test that the matrices in the models do what we intend.

```
sage: from sage.geometry.hyperbolic_space.hyperbolic_isometry \
....:   import moebius_transform
sage: R = H.PD().get_geodesic(-1,1).reflection_involution()
sage: bool(moebius_transform(R.matrix(), 0) == 0)
True
```

```
>>> from sage.all import *
>>> from sage.geometry.hyperbolic_space.hyperbolic_isometry   import moebius_
↪transform
>>> R = H.PD().get_geodesic(-Integer(1),Integer(1)).reflection_involution()
>>> bool(moebius_transform(R.matrix(), Integer(0)) == Integer(0))
```

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```
True
```

start()

Return the starting point of the geodesic.

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(I, 3*I)
sage: g.start()
Point in UHP I
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(I, Integer(3)*I)
>>> g.start()
Point in UHP I
```

to_model(*model*)

Convert the current object to image in another model.

INPUT:

- `model` – the image model

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: PD = HyperbolicPlane().PD()
sage: UHP.get_geodesic(I, 2*I).to_model(PD)
Geodesic in PD from 0 to 1/3*I
sage: UHP.get_geodesic(I, 2*I).to_model('PD')
Geodesic in PD from 0 to 1/3*I
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> PD = HyperbolicPlane().PD()
>>> UHP.get_geodesic(I, Integer(2)*I).to_model(PD)
Geodesic in PD from 0 to 1/3*I
>>> UHP.get_geodesic(I, Integer(2)*I).to_model('PD')
Geodesic in PD from 0 to 1/3*I
```

update_graphics(*update=False*, ***options*)

Update the graphics options of `self`.

INPUT:

- `update` – if `True`, the original option are updated rather than overwritten

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(I, 2*I)
sage: g.graphics_options()
{}

sage: g.update_graphics(color = "red"); g.graphics_options()
{'color': 'red'}
```

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```
sage: g.update_graphics(color = "blue"); g.graphics_options()
{'color': 'blue'}

sage: g.update_graphics(True, size = 20); g.graphics_options()
{'color': 'blue', 'size': 20}
```

```
>>> from sage.all import *
>>> g = HyperbolicPlane().UHP().get_geodesic(I, Integer(2)*I)
>>> g.graphics_options()
{}

>>> g.update_graphics(color = "red"); g.graphics_options()
{'color': 'red'}

>>> g.update_graphics(color = "blue"); g.graphics_options()
{'color': 'blue'}

>>> g.update_graphics(True, size = Integer(20)); g.graphics_options()
{'color': 'blue', 'size': 20}
```

class sage.geometry.hyperbolic_space.hyperbolic_geodesic.**HyperbolicGeodesicHM**(*model*, *start*, *end*, ***graphics_options*)

Bases: *HyperbolicGeodesic*

A geodesic in the hyperboloid model.

Valid points in the hyperboloid model satisfy $x^2 + y^2 - z^2 = -1$

INPUT:

- `start` – a `HyperbolicPoint` in hyperbolic space representing the start of the geodesic
- `end` – a `HyperbolicPoint` in hyperbolic space representing the end of the geodesic

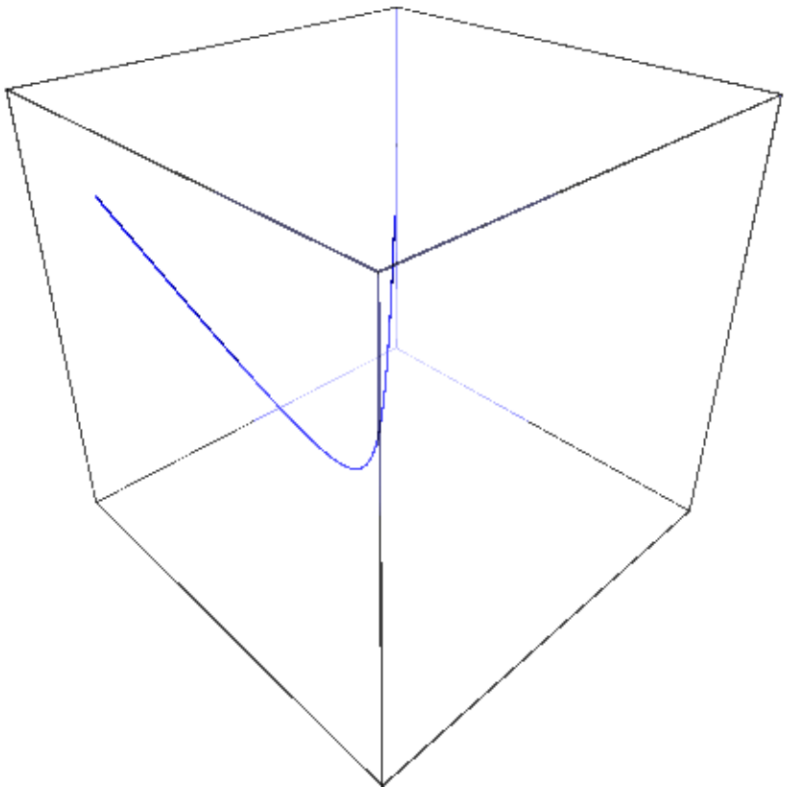
EXAMPLES:

```
sage: HM = HyperbolicPlane().HM()
sage: p1 = HM.get_point((4, -4, sqrt(33)))
sage: p2 = HM.get_point((-3,-3,sqrt(19)))
sage: g = HM.get_geodesic(p1, p2)
sage: g = HM.get_geodesic((4, -4, sqrt(33)), (-3, -3, sqrt(19)))
```

```
>>> from sage.all import *
>>> HM = HyperbolicPlane().HM()
>>> p1 = HM.get_point((Integer(4), -Integer(4), sqrt(Integer(33))))
>>> p2 = HM.get_point((-Integer(3),-Integer(3),sqrt(Integer(19))))
>>> g = HM.get_geodesic(p1, p2)
>>> g = HM.get_geodesic((Integer(4), -Integer(4), sqrt(Integer(33))), (-
↪Integer(3), -Integer(3), sqrt(Integer(19))))
```

plot(*show_hyperboloid=True*, ***graphics_options*)

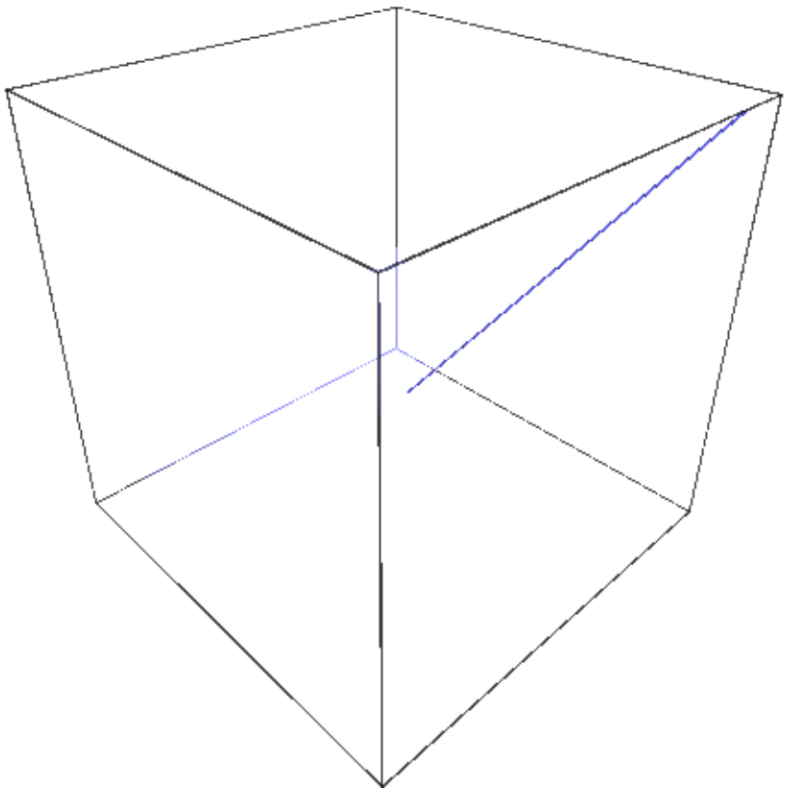
Plot `self`.



EXAMPLES:

```
sage: from sage.geometry.hyperbolic_space.hyperbolic_geodesic \
....:     import *
sage: g = HyperbolicPlane().HM().random_geodesic()
sage: g.plot()                                                              #␣
↪needs sage.plot
Graphics3d Object
```

```
>>> from sage.all import *
>>> from sage.geometry.hyperbolic_space.hyperbolic_geodesic     import *
>>> g = HyperbolicPlane().HM().random_geodesic()
>>> g.plot()                                                                #␣
↪needs sage.plot
Graphics3d Object
```



class sage.geometry.hyperbolic_space.hyperbolic_geodesic.**HyperbolicGeodesicKM**(*model*, *start*, *end*, ***graphics_options*)

Bases: *HyperbolicGeodesic*

A geodesic in the Klein disk model.

Geodesics are represented by the chords, straight line segments with ideal endpoints on the boundary circle.

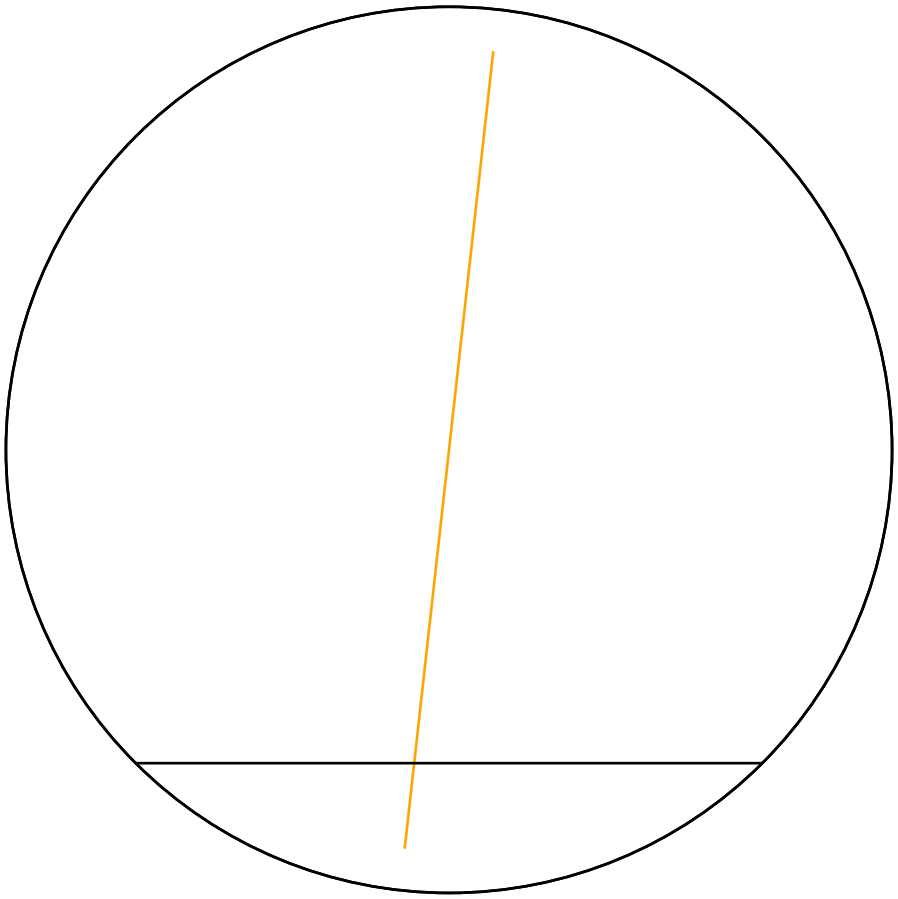
INPUT:

- `start` – a `HyperbolicPoint` in hyperbolic space representing the start of the geodesic
- `end` – a `HyperbolicPoint` in hyperbolic space representing the end of the geodesic

EXAMPLES:

```
sage: KM = HyperbolicPlane().KM()
sage: g = KM.get_geodesic((0.1,0.9),(-0.1,-0.9))
sage: h = KM.get_geodesic((-0.707106781,-0.707106781),(0.707106781,-0.707106781))
sage: P = g.plot(color='orange')+h.plot(); P                                    #␣
↪needs sage.plot
Graphics object consisting of 4 graphics primitives
```

```
>>> from sage.all import *
>>> KM = HyperbolicPlane().KM()
>>> g = KM.get_geodesic((RealNumber('0.1'),RealNumber('0.9')),(-RealNumber('0.1'),
↪-RealNumber('0.9')))
>>> h = KM.get_geodesic((-RealNumber('0.707106781'),-RealNumber('0.707106781')),
↪(RealNumber('0.707106781'),-RealNumber('0.707106781')))
>>> P = g.plot(color='orange')+h.plot(); P                                      #␣
↪needs sage.plot
Graphics object consisting of 4 graphics primitives
```



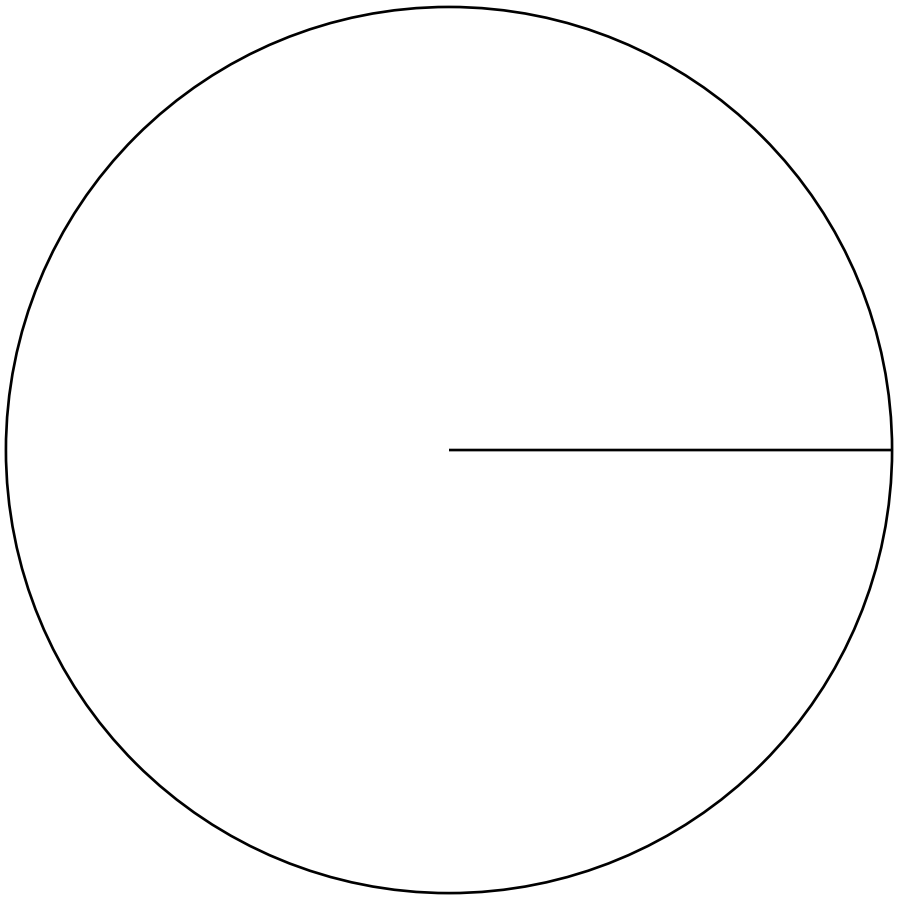
plot(*boundary=True*, ***options*)

Plot `self`.

EXAMPLES:

```
sage: HyperbolicPlane().KM().get_geodesic(0, 1).plot()                    #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

```
>>> from sage.all import *
>>> HyperbolicPlane().KM().get_geodesic(Integer(0), Integer(1)).plot()       ␣
↪              # needs sage.plot
Graphics object consisting of 2 graphics primitives
```



class sage.geometry.hyperbolic_space.hyperbolic_geodesic.**HyperbolicGeodesicPD**(*model*, *start*, *end*, ***graphics_options*)

Bases: *HyperbolicGeodesic*

A geodesic in the Poincaré disk model.

Geodesics in this model are represented by segments of circles contained within the unit disk that are orthogonal to the boundary of the disk, plus all diameters of the disk.

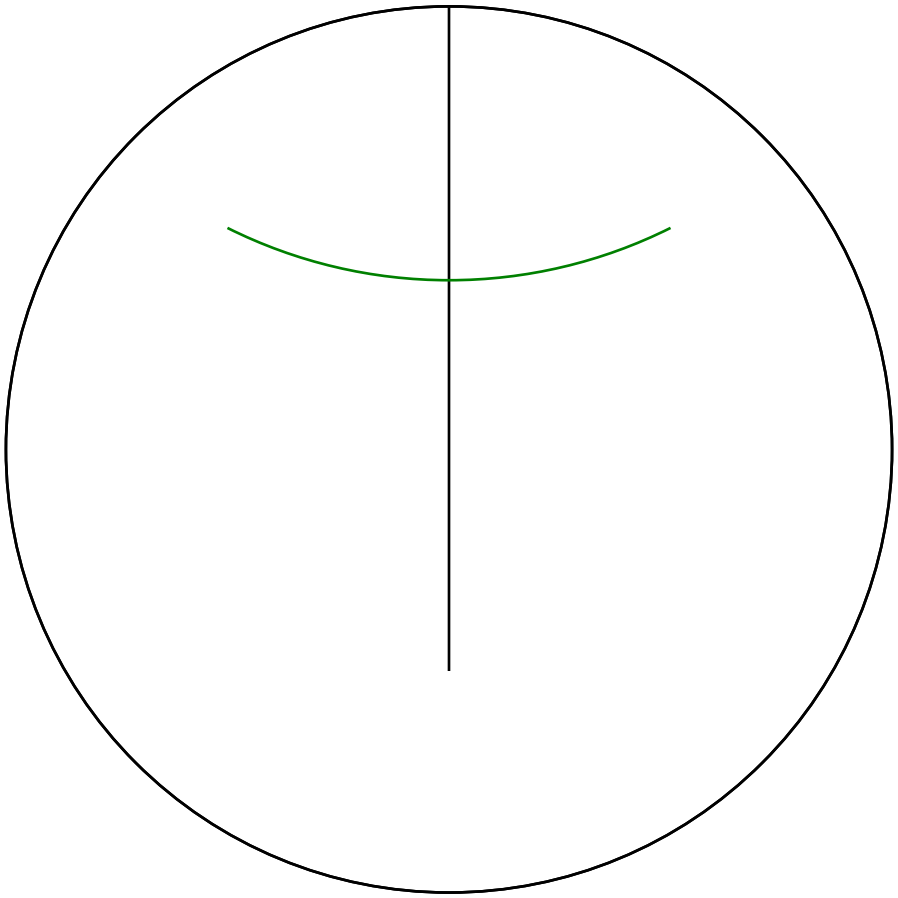
INPUT:

- `start` – a `HyperbolicPoint` in hyperbolic space representing the start of the geodesic
- `end` – a `HyperbolicPoint` in hyperbolic space representing the end of the geodesic

EXAMPLES:

```
sage: PD = HyperbolicPlane().PD()
sage: g = PD.get_geodesic(PD.get_point(I), PD.get_point(-I/2))
sage: g = PD.get_geodesic(I,-I/2)
sage: h = PD.get_geodesic(-1/2+I/2,1/2+I/2)
```

```
>>> from sage.all import *
>>> PD = HyperbolicPlane().PD()
>>> g = PD.get_geodesic(PD.get_point(I), PD.get_point(-I/Integer(2)))
>>> g = PD.get_geodesic(I,-I/Integer(2))
>>> h = PD.get_geodesic(-Integer(1)/Integer(2)+I/Integer(2),Integer(1)/
↪Integer(2)+I/Integer(2))
```



plot(*boundary=True*, ***options*)

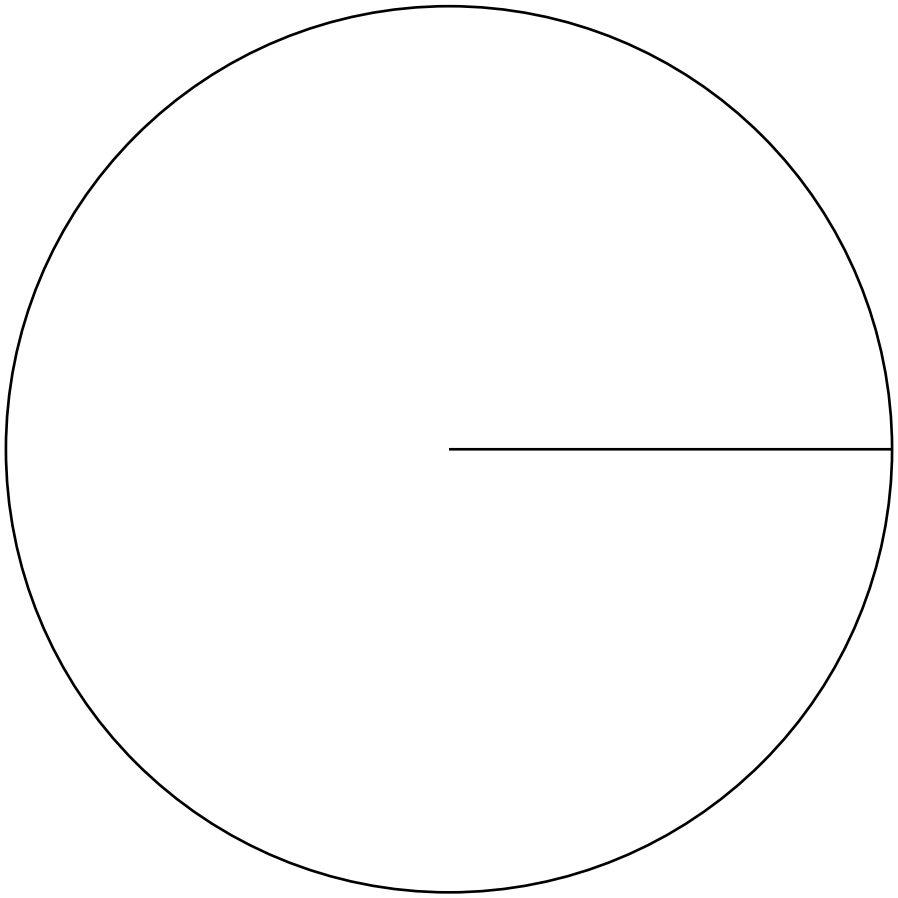
Plot `self`.

EXAMPLES:

First some lines:

```
sage: PD = HyperbolicPlane().PD()
sage: PD.get_geodesic(0, 1).plot()                                              #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

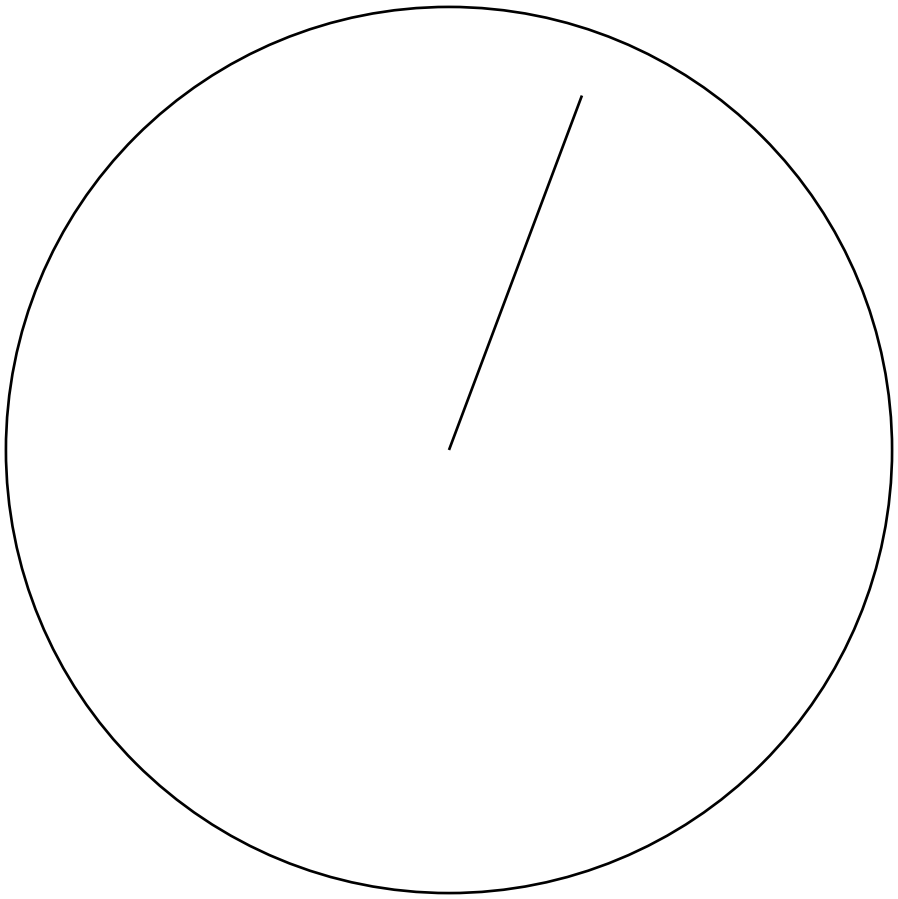
```
>>> from sage.all import *
>>> PD = HyperbolicPlane().PD()
>>> PD.get_geodesic(Integer(0), Integer(1)).plot()                              ␣
↪              # needs sage.plot
Graphics object consisting of 2 graphics primitives
```



```
sage: PD.get_geodesic(0, 0.3+0.8*I).plot()                                      #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

```
>>> from sage.all import *
>>> PD.get_geodesic(Integer(0), RealNumber('0.3')+RealNumber('0.8')*I).plot()␣
↪                                  # needs sage.plot
Graphics object consisting of 2 graphics primitives
```

Then some generic geodesics:



```
sage: PD.get_geodesic(-0.5, 0.3+0.4*I).plot()                                  #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
sage: g = PD.get_geodesic(-1, exp(3*I*pi/7))
sage: G = g.plot(linestyle='dashed',color='red'); G                            #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
sage: h = PD.get_geodesic(exp(2*I*pi/11), exp(1*I*pi/11))
sage: H = h.plot(thickness=6, color='orange'); H                               #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
sage: show(G+H)                                                                #␣
↪needs sage.plot
```

```
>>> from sage.all import *
>>> PD.get_geodesic(-RealNumber('0.5'), RealNumber('0.3')+RealNumber('0.4
↪')*I).plot()                                  # needs sage.plot
Graphics object consisting of 2 graphics primitives
>>> g = PD.get_geodesic(-Integer(1), exp(Integer(3)*I*pi/Integer(7)))
>>> G = g.plot(linestyle='dashed',color='red'); G                           #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
>>> h = PD.get_geodesic(exp(Integer(2)*I*pi/Integer(11)), exp(Integer(1)*I*pi/
↪Integer(11)))
>>> H = h.plot(thickness=Integer(6), color='orange'); H                        ␣
↪     # needs sage.plot
Graphics object consisting of 2 graphics primitives
>>> show(G+H)                                                                #␣
↪needs sage.plot
```

class sage.geometry.hyperbolic_space.hyperbolic_geodesic.**HyperbolicGeodesicUHP**(*model*, *start*, *end*, ***graphics_options*)

Bases: *HyperbolicGeodesic*

Create a geodesic in the upper half plane model.

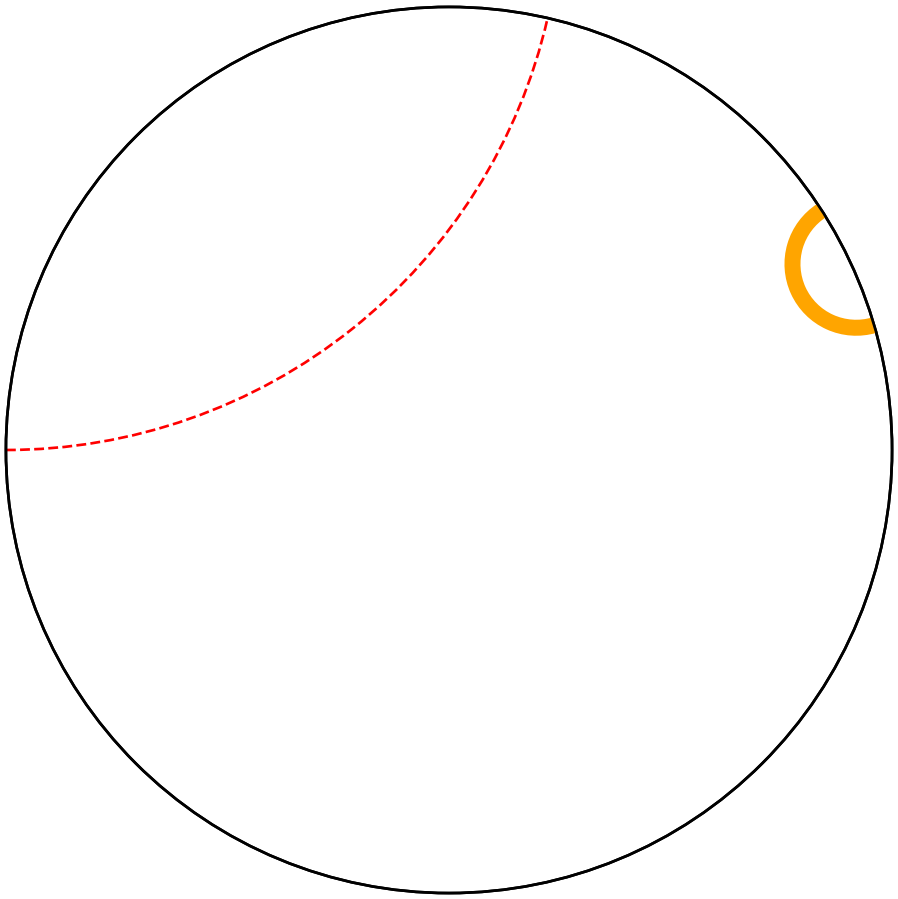
The geodesics in this model are represented by circular arcs perpendicular to the real axis (half-circles whose origin is on the real axis) and straight vertical lines ending on the real axis.

INPUT:

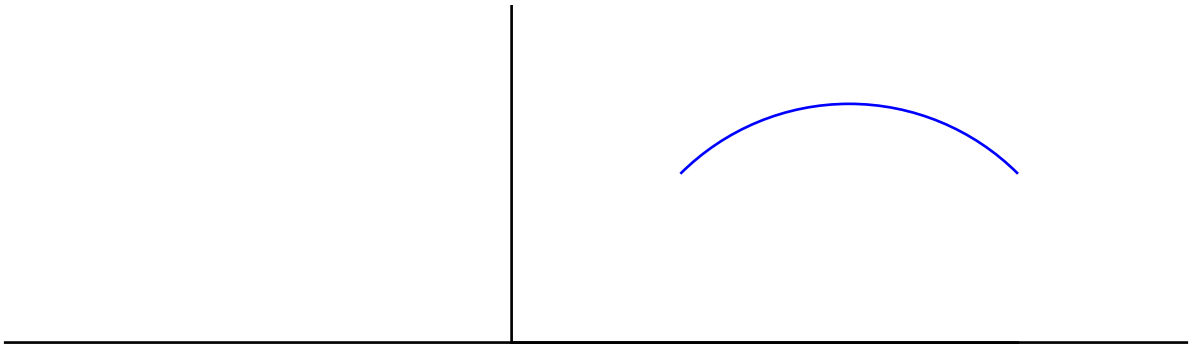
- `start` – a `HyperbolicPoint` in hyperbolic space representing the start of the geodesic
- `end` – a `HyperbolicPoint` in hyperbolic space representing the end of the geodesic

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: g = UHP.get_geodesic(UHP.get_point(I), UHP.get_point(2 + I))
sage: g = UHP.get_geodesic(I, 2 + I)
sage: h = UHP.get_geodesic(-1, -1+2*I)
```



```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> g = UHP.get_geodesic(UHP.get_point(I), UHP.get_point(Integer(2) + I))
>>> g = UHP.get_geodesic(I, Integer(2) + I)
>>> h = UHP.get_geodesic(-Integer(1), -Integer(1)+Integer(2)*I)
```



angle(*other*)

Return the angle between the completions of any two given geodesics if they intersect.

INPUT:

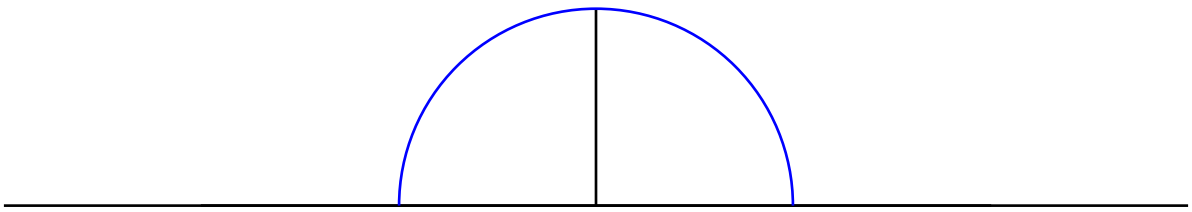
- `other` – a hyperbolic geodesic in the UHP model

OUTPUT: the angle in radians between the two given geodesics

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: g = UHP.get_geodesic(2, 4)
sage: h = UHP.get_geodesic(3, 3 + I)
sage: g.angle(h)
1/2*pi
sage: numerical_approx(g.angle(h))
1.57079632679490
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> g = UHP.get_geodesic(Integer(2), Integer(4))
>>> h = UHP.get_geodesic(Integer(3), Integer(3) + I)
>>> g.angle(h)
1/2*pi
>>> numerical_approx(g.angle(h))
1.57079632679490
```



If the geodesics are identical, return angle 0:

```
sage: g.angle(g)
0
```

```
>>> from sage.all import *
>>> g.angle(g)
0
```

It is an error to ask for the angle of two geodesics that do not intersect:

```
sage: g = UHP.get_geodesic(2, 4)
sage: h = UHP.get_geodesic(5, 7)
sage: g.angle(h)
```

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```
Traceback (most recent call last):
...
ValueError: geodesics do not intersect
```

```
>>> from sage.all import *
>>> g = UHP.get_geodesic(Integer(2), Integer(4))
>>> h = UHP.get_geodesic(Integer(5), Integer(7))
>>> g.angle(h)
Traceback (most recent call last):
...
ValueError: geodesics do not intersect
```

common_perpendicular(*other*)

Return the unique hyperbolic geodesic perpendicular to `self` and `other`, if such a geodesic exists; otherwise raise a `ValueError`.

INPUT:

- `other` – a hyperbolic geodesic in current model

OUTPUT: a hyperbolic geodesic

EXAMPLES:

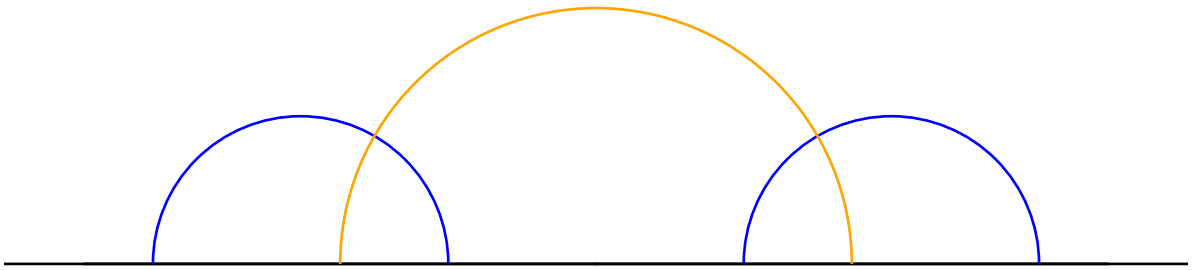
```
sage: UHP = HyperbolicPlane().UHP()
sage: g = UHP.get_geodesic(2, 3)
sage: h = UHP.get_geodesic(4, 5)
sage: g.common_perpendicular(h)
Geodesic in UHP from 1/2*sqrt(3) + 7/2 to -1/2*sqrt(3) + 7/2
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> g = UHP.get_geodesic(Integer(2), Integer(3))
>>> h = UHP.get_geodesic(Integer(4), Integer(5))
>>> g.common_perpendicular(h)
Geodesic in UHP from 1/2*sqrt(3) + 7/2 to -1/2*sqrt(3) + 7/2
```

It is an error to ask for the common perpendicular of two intersecting geodesics:

```
sage: g = UHP.get_geodesic(2, 4)
sage: h = UHP.get_geodesic(3, infinity)
sage: g.common_perpendicular(h)
Traceback (most recent call last):
...
ValueError: geodesics intersect; no common perpendicular exists
```

```
>>> from sage.all import *
>>> g = UHP.get_geodesic(Integer(2), Integer(4))
>>> h = UHP.get_geodesic(Integer(3), infinity)
>>> g.common_perpendicular(h)
Traceback (most recent call last):
...
ValueError: geodesics intersect; no common perpendicular exists
```



ideal_endpoints()

Determine the ideal (boundary) endpoints of the complete hyperbolic geodesic corresponding to `self`.

OUTPUT: list of 2 boundary points

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_geodesic(I, 2*I).ideal_endpoints()
[Boundary point in UHP 0,
 Boundary point in UHP +Infinity]
sage: UHP.get_geodesic(1 + I, 2 + 4*I).ideal_endpoints()
[Boundary point in UHP -sqrt(65) + 9,
 Boundary point in UHP sqrt(65) + 9]
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.get_geodesic(I, Integer(2)*I).ideal_endpoints()
[Boundary point in UHP 0,
 Boundary point in UHP +Infinity]
>>> UHP.get_geodesic(Integer(1) + I, Integer(2) + Integer(4)*I).ideal_
↪endpoints()
[Boundary point in UHP -sqrt(65) + 9,
 Boundary point in UHP sqrt(65) + 9]
```

intersection(*other*)

Return the point of intersection of `self` and `other` (if such a point exists).

INPUT:

- `other` – a hyperbolic geodesic in the current model

OUTPUT: list of hyperbolic points or a hyperbolic geodesic

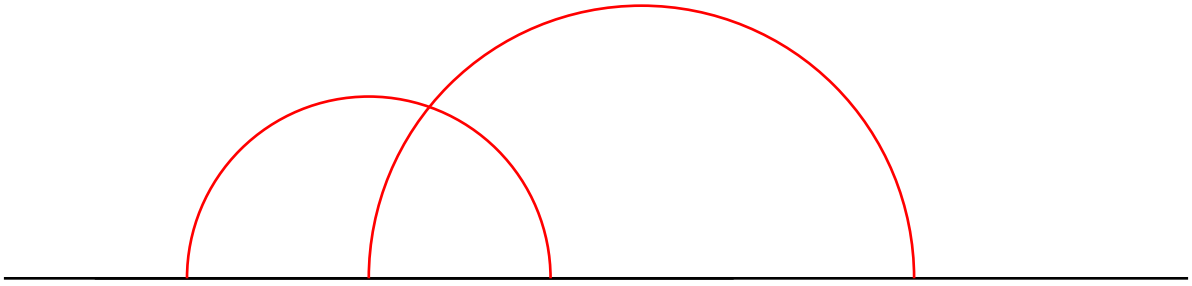
EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: g = UHP.get_geodesic(3, 5)
sage: h = UHP.get_geodesic(4, 7)
sage: g.intersection(h)
[Point in UHP 2/3*sqrt(-2) + 13/3]
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> g = UHP.get_geodesic(Integer(3), Integer(5))
>>> h = UHP.get_geodesic(Integer(4), Integer(7))
>>> g.intersection(h)
[Point in UHP 2/3*sqrt(-2) + 13/3]
```

If the given geodesics do not intersect, the function returns an empty list:

```
sage: g = UHP.get_geodesic(4, 5)
sage: h = UHP.get_geodesic(6, 7)
sage: g.intersection(h)
[]
```



```
>>> from sage.all import *
>>> g = UHP.get_geodesic(Integer(4), Integer(5))
>>> h = UHP.get_geodesic(Integer(6), Integer(7))
>>> g.intersection(h)
[]
```

If the given geodesics are asymptotically parallel, the function returns the common boundary point:

```
sage: g = UHP.get_geodesic(4, 5)
sage: h = UHP.get_geodesic(5, 7)
sage: g.intersection(h)
[Boundary point in UHP 5.000000000000000]
```

```
>>> from sage.all import *
>>> g = UHP.get_geodesic(Integer(4), Integer(5))
>>> h = UHP.get_geodesic(Integer(5), Integer(7))
>>> g.intersection(h)
[Boundary point in UHP 5.000000000000000]
```

If the given geodesics are identical, return that geodesic:

```
sage: g = UHP.get_geodesic(4 + I, 18*I)
sage: h = UHP.get_geodesic(4 + I, 18*I)
```

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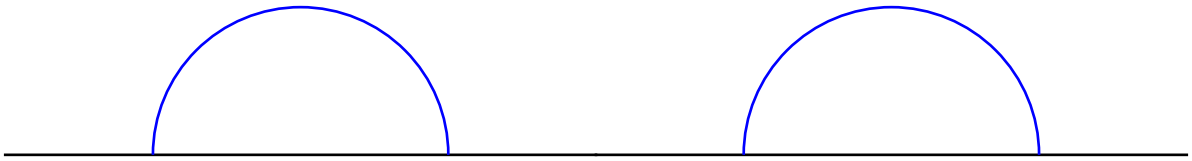

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```
sage: g.intersection(h)
Geodesic in UHP from I + 4 to 18*I
```

```
>>> from sage.all import *
>>> g = UHP.get_geodesic(Integer(4) + I, Integer(18)*I)
>>> h = UHP.get_geodesic(Integer(4) + I, Integer(18)*I)
>>> g.intersection(h)
Geodesic in UHP from I + 4 to 18*I
```

midpoint()

Return the (hyperbolic) midpoint of `self` if it exists.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: g = UHP.random_geodesic()
sage: m = g.midpoint()
sage: d1 = UHP.dist(m, g.start())
sage: d2 = UHP.dist(m, g.end())
sage: bool(abs(d1 - d2) < 10**-9)
True
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> g = UHP.random_geodesic()
>>> m = g.midpoint()
>>> d1 = UHP.dist(m, g.start())
>>> d2 = UHP.dist(m, g.end())
>>> bool(abs(d1 - d2) < Integer(10)**-Integer(9))
True
```

Infinite geodesics have no midpoint:

```
sage: UHP.get_geodesic(0, 2).midpoint()
Traceback (most recent call last):
...
ValueError: the length must be finite
```

```
>>> from sage.all import *
>>> UHP.get_geodesic(Integer(0), Integer(2)).midpoint()
Traceback (most recent call last):
...
ValueError: the length must be finite
```

perpendicular_bisector()

Return the perpendicular bisector of the hyperbolic geodesic `self` if that geodesic has finite length.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: g = UHP.random_geodesic()
sage: h = g.perpendicular_bisector().complete()
```

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```
sage: c = lambda x: x.coordinates()
sage: bool(c(g.intersection(h)[0]) - c(g.midpoint()) < 10**-9)
True
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> g = UHP.random_geodesic()
>>> h = g.perpendicular_bisector().complete()
>>> c = lambda x: x.coordinates()
>>> bool(c(g.intersection(h)[Integer(0)]) - c(g.midpoint()) < Integer(10)**-
↪Integer(9))
True
```

```
sage: UHP = HyperbolicPlane().UHP()
sage: g = UHP.get_geodesic(1+I,2+0.5*I)
sage: h = g.perpendicular_bisector().complete()
sage: show(g.plot(color='blue')+h.plot(color='orange'))
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> g = UHP.get_geodesic(Integer(1)+I,Integer(2)+RealNumber('0.5')*I)
>>> h = g.perpendicular_bisector().complete()
>>> show(g.plot(color='blue')+h.plot(color='orange'))
```

Infinite geodesics cannot be bisected:

```
sage: UHP.get_geodesic(0, 1).perpendicular_bisector()
Traceback (most recent call last):
...
ValueError: the length must be finite
```

```
>>> from sage.all import *
>>> UHP.get_geodesic(Integer(0), Integer(1)).perpendicular_bisector()
Traceback (most recent call last):
...
ValueError: the length must be finite
```

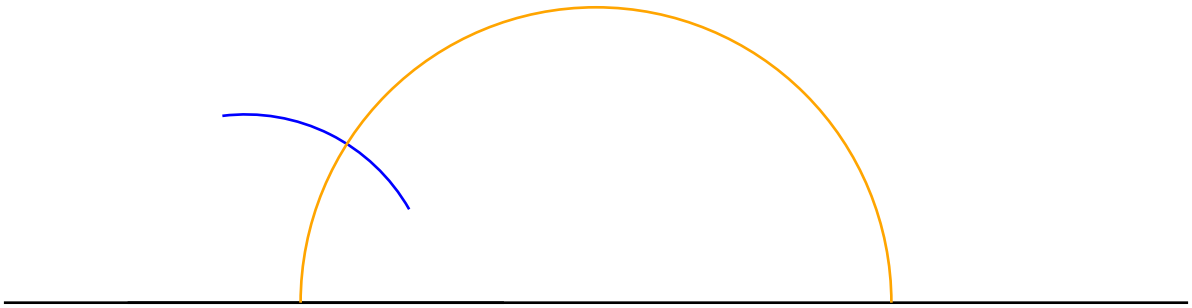
plot(*boundary=True*, ***options*)

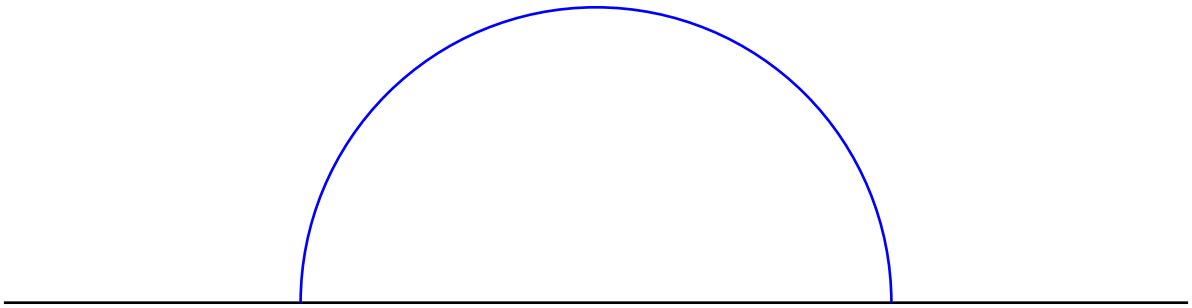
Plot `self`.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_geodesic(0, 1).plot()                                             #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

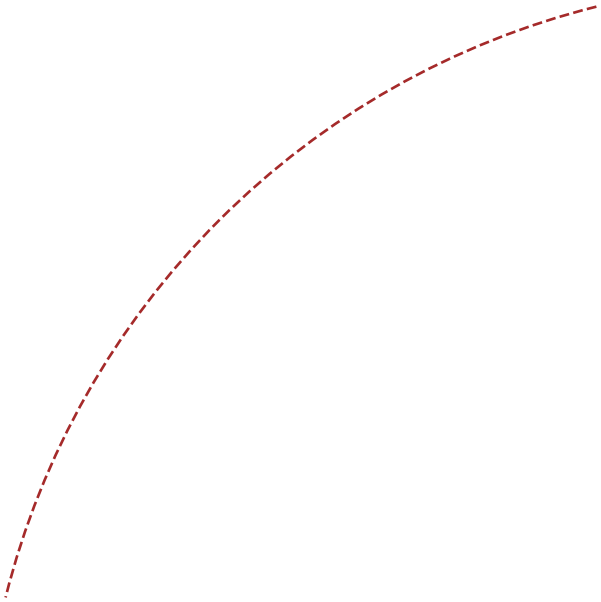
```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.get_geodesic(Integer(0), Integer(1)).plot()                             ␣
↪              # needs sage.plot
Graphics object consisting of 2 graphics primitives
```





```
sage: UHP.get_geodesic(I, 3+4*I).plot(linestyle='dashed', color='brown')     #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

```
>>> from sage.all import *
>>> UHP.get_geodesic(I, Integer(3)+Integer(4)*I).plot(linestyle='dashed',␣
↪color='brown')     # needs sage.plot
Graphics object consisting of 2 graphics primitives
```



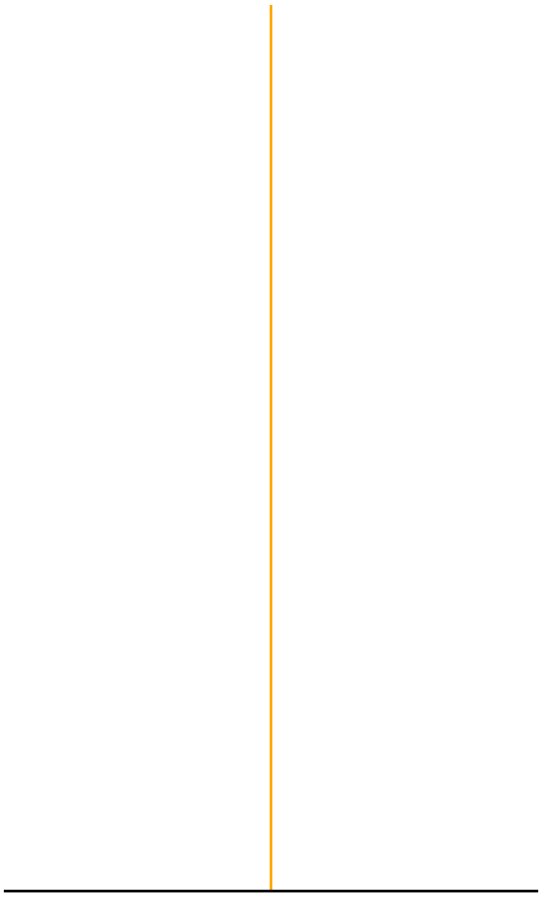
```
sage: UHP.get_geodesic(1, infinity).plot(color='orange')                     #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

```
>>> from sage.all import *
>>> UHP.get_geodesic(Integer(1), infinity).plot(color='orange')              ␣
↪     # needs sage.plot
Graphics object consisting of 2 graphics primitives
```

reflection_involution()

Return the isometry of the involution fixing the geodesic `self`.

EXAMPLES:



```
sage: UHP = HyperbolicPlane().UHP()
sage: g1 = UHP.get_geodesic(0, 1)
sage: g1.reflection_involution()
Isometry in UHP
[ 1  0]
[ 2 -1]
sage: UHP.get_geodesic(I, 2*I).reflection_involution()
Isometry in UHP
[ 1  0]
[ 0 -1]
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> g1 = UHP.get_geodesic(Integer(0), Integer(1))
>>> g1.reflection_involution()
Isometry in UHP
[ 1  0]
[ 2 -1]
>>> UHP.get_geodesic(I, Integer(2)*I).reflection_involution()
Isometry in UHP
[ 1  0]
[ 0 -1]
```

CHAPTER

FOUR

HYPERBOLIC MODELS

In this module, a hyperbolic model is a collection of data that allow the user to implement new models of hyperbolic space with minimal effort. The data include facts about the underlying set (such as whether the model is bounded), facts about the metric (such as whether the model is conformal), facts about the isometry group (such as whether it is a linear or projective group), and more. Generally speaking, any data or method that pertains to the model itself – rather than the points, geodesics, or isometries of the model – is implemented in this module.

Abstractly, a model of hyperbolic space is a connected, simply connected manifold equipped with a complete Riemannian metric of constant curvature -1. This module records information sufficient to enable computations in hyperbolic space without explicitly specifying the underlying set or its Riemannian metric. Although, see the SageManifolds project if you would like to take this approach.

This module implements the abstract base class for a model of hyperbolic space of arbitrary dimension. It also contains the implementations of specific models of hyperbolic geometry.

AUTHORS:

- Greg Laun (2013): Initial version.

EXAMPLES:

We illustrate how the classes in this module encode data by comparing the upper half plane (UHP), Poincaré disk (PD) and hyperboloid (HM) models. First we create:

```
sage: U = HyperbolicPlane().UHP()
sage: P = HyperbolicPlane().PD()
sage: H = HyperbolicPlane().HM()
```

```
>>> from sage.all import *
>>> U = HyperbolicPlane().UHP()
>>> P = HyperbolicPlane().PD()
>>> H = HyperbolicPlane().HM()
```

We note that the UHP and PD models are bounded while the HM model is not:

```
sage: U.is_bounded() and P.is_bounded()
True
sage: H.is_bounded()
False
```

```
>>> from sage.all import *
>>> U.is_bounded() and P.is_bounded()
True
```

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```
>>> H.is_bounded()
False
```

The isometry groups of UHP and PD are projective, while that of HM is linear:

```
sage: U.is_isometry_group_projective()
True
sage: H.is_isometry_group_projective()
False
```

```
>>> from sage.all import *
>>> U.is_isometry_group_projective()
True
>>> H.is_isometry_group_projective()
False
```

The models are responsible for determining if the coordinates of points and the matrix of linear maps are appropriate for constructing points and isometries in hyperbolic space:

```
sage: U.point_in_model(2 + I)
True
sage: U.point_in_model(2 - I)
False
sage: U.point_in_model(2)
False
sage: U.boundary_point_in_model(2)
True
```

```
>>> from sage.all import *
>>> U.point_in_model(Integer(2) + I)
True
>>> U.point_in_model(Integer(2) - I)
False
>>> U.point_in_model(Integer(2))
False
>>> U.boundary_point_in_model(Integer(2))
True
```

class sage.geometry.hyperbolic_space.hyperbolic_model.**HyperbolicModel**(*space*, *name*, *short_name*, *bounded*, *conformal*, *dimension*, *isometry_group*, *isometry_group_is_projective*)

Bases: `Parent`, `UniqueRepresentation`, `BindableClass`

Abstract base class for hyperbolic models.

Element

alias of *HyperbolicPoint*

bdry_point_test(*p*)

Test whether a point is in the model. If the point is in the model, do nothing; otherwise raise a `ValueError`.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().bdry_point_test(2)
sage: HyperbolicPlane().UHP().bdry_point_test(1 + I)
Traceback (most recent call last):
...
ValueError: I + 1 is not a valid boundary point in the UHP model
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().bdry_point_test(Integer(2))
>>> HyperbolicPlane().UHP().bdry_point_test(Integer(1) + I)
Traceback (most recent call last):
...
ValueError: I + 1 is not a valid boundary point in the UHP model
```

boundary_point_in_model(*p*)

Return `True` if the point is on the ideal boundary of hyperbolic space and `False` otherwise.

INPUT:

- any object that can converted into a complex number

OUTPUT: boolean

EXAMPLES:

```
sage: HyperbolicPlane().UHP().boundary_point_in_model(I)
False
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().boundary_point_in_model(I)
False
```

dist(*a*, *b*)

Calculate the hyperbolic distance between `a` and `b`.

INPUT:

- `a`, `b` – a point or geodesic

OUTPUT: the hyperbolic distance

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: p1 = UHP.get_point(5 + 7*I)
sage: p2 = UHP.get_point(1.0 + I)
sage: UHP.dist(p1, p2)
2.23230104635820

sage: PD = HyperbolicPlane().PD()
sage: p1 = PD.get_point(0)
sage: p2 = PD.get_point(I/2)
sage: PD.dist(p1, p2)
arccosh(5/3)

sage: UHP(p1).dist(UHP(p2))
```

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```
arccosh(5/3)

sage: KM = HyperbolicPlane().KM()
sage: p1 = KM.get_point((0, 0))
sage: p2 = KM.get_point((1/2, 1/2))
sage: numerical_approx(KM.dist(p1, p2))
0.881373587019543

sage: HM = HyperbolicPlane().HM()
sage: p1 = HM.get_point((0,0,1))
sage: p2 = HM.get_point((1,0,sqrt(2)))
sage: numerical_approx(HM.dist(p1, p2))
0.881373587019543
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> p1 = UHP.get_point(Integer(5) + Integer(7)*I)
>>> p2 = UHP.get_point(RealNumber('1.0') + I)
>>> UHP.dist(p1, p2)
2.23230104635820

>>> PD = HyperbolicPlane().PD()
>>> p1 = PD.get_point(Integer(0))
>>> p2 = PD.get_point(I/Integer(2))
>>> PD.dist(p1, p2)
arccosh(5/3)

>>> UHP(p1).dist(UHP(p2))
arccosh(5/3)

>>> KM = HyperbolicPlane().KM()
>>> p1 = KM.get_point((Integer(0), Integer(0)))
>>> p2 = KM.get_point((Integer(1)/Integer(2), Integer(1)/Integer(2)))
>>> numerical_approx(KM.dist(p1, p2))
0.881373587019543

>>> HM = HyperbolicPlane().HM()
>>> p1 = HM.get_point((Integer(0),Integer(0),Integer(1)))
>>> p2 = HM.get_point((Integer(1),Integer(0),sqrt(Integer(2))))
>>> numerical_approx(HM.dist(p1, p2))
0.881373587019543
```

Distance between a point and itself is 0:

```
sage: p = UHP.get_point(47 + I)
sage: UHP.dist(p, p)
0
```

```
>>> from sage.all import *
>>> p = UHP.get_point(Integer(47) + I)
>>> UHP.dist(p, p)
0
```

Points on the boundary are infinitely far from interior points:

```
sage: UHP.get_point(3).dist(UHP.get_point(I))
+Infinity
```

```
>>> from sage.all import *
>>> UHP.get_point(Integer(3)).dist(UHP.get_point(I))
+Infinity
```

get_geodesic(*start*, *end=None*, ***graphics_options*)

Return a geodesic in the appropriate model.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_geodesic(I, 2*I)
Geodesic in UHP from I to 2*I

sage: HyperbolicPlane().PD().get_geodesic(0, I/2)
Geodesic in PD from 0 to 1/2*I

sage: HyperbolicPlane().KM().get_geodesic((1/2, 1/2), (0,0))
Geodesic in KM from (1/2, 1/2) to (0, 0)

sage: HyperbolicPlane().HM().get_geodesic((0,0,1), (1,0, sqrt(2)))
Geodesic in HM from (0, 0, 1) to (1, 0, sqrt(2))
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_geodesic(I, Integer(2)*I)
Geodesic in UHP from I to 2*I

>>> HyperbolicPlane().PD().get_geodesic(Integer(0), I/Integer(2))
Geodesic in PD from 0 to 1/2*I

>>> HyperbolicPlane().KM().get_geodesic((Integer(1)/Integer(2), Integer(1)/
↪Integer(2)), (Integer(0),Integer(0)))
Geodesic in KM from (1/2, 1/2) to (0, 0)

>>> HyperbolicPlane().HM().get_geodesic((Integer(0),Integer(0),Integer(1)),␣
↪(Integer(1),Integer(0), sqrt(Integer(2))))
Geodesic in HM from (0, 0, 1) to (1, 0, sqrt(2))
```

get_isometry(*A*)

Return an isometry in `self` from the matrix `A` in the isometry group of `self`.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_isometry(identity_matrix(2))
Isometry in UHP
[1 0]
[0 1]

sage: HyperbolicPlane().PD().get_isometry(identity_matrix(2))
Isometry in PD
[1 0]
```

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```
[0 1]

sage: HyperbolicPlane().KM().get_isometry(identity_matrix(3))                #␣
↪needs scipy
Isometry in KM
[1 0 0]
[0 1 0]
[0 0 1]

sage: HyperbolicPlane().HM().get_isometry(identity_matrix(3))                #␣
↪needs scipy
Isometry in HM
[1 0 0]
[0 1 0]
[0 0 1]
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_isometry(identity_matrix(Integer(2)))
Isometry in UHP
[1 0]
[0 1]

>>> HyperbolicPlane().PD().get_isometry(identity_matrix(Integer(2)))
Isometry in PD
[1 0]
[0 1]

>>> HyperbolicPlane().KM().get_isometry(identity_matrix(Integer(3)))          ␣
↪     # needs scipy
Isometry in KM
[1 0 0]
[0 1 0]
[0 0 1]

>>> HyperbolicPlane().HM().get_isometry(identity_matrix(Integer(3)))          ␣
↪     # needs scipy
Isometry in HM
[1 0 0]
[0 1 0]
[0 0 1]
```

get_point(*coordinates*, *is_boundary=None*, ***graphics_options*)

Return a point in `self`.

Automatically determine the type of point to return given either:

1. the coordinates of a point in the interior or ideal boundary of hyperbolic space, or
2. a *HyperbolicPoint* object.

INPUT:

- a point in hyperbolic space or on the ideal boundary

OUTPUT: a *HyperbolicPoint*

EXAMPLES:

We can create an interior point via the coordinates:

```
sage: HyperbolicPlane().UHP().get_point(2*I)
Point in UHP 2*I
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_point(Integer(2)*I)
Point in UHP 2*I
```

Or we can create a boundary point via the coordinates:

```
sage: HyperbolicPlane().UHP().get_point(23)
Boundary point in UHP 23
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_point(Integer(23))
Boundary point in UHP 23
```

However we cannot create points outside of our model:

```
sage: HyperbolicPlane().UHP().get_point(12 - I)
Traceback (most recent call last):
...
ValueError: -I + 12 is not a valid point in the UHP model
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_point(Integer(12) - I)
Traceback (most recent call last):
...
ValueError: -I + 12 is not a valid point in the UHP model
```

```
sage: HyperbolicPlane().UHP().get_point(2 + 3*I)
Point in UHP 3*I + 2

sage: HyperbolicPlane().PD().get_point(0)
Point in PD 0

sage: HyperbolicPlane().KM().get_point((0,0))
Point in KM (0, 0)

sage: HyperbolicPlane().HM().get_point((0,0,1))
Point in HM (0, 0, 1)

sage: p = HyperbolicPlane().UHP().get_point(I, color='red')
sage: p.graphics_options()
{'color': 'red'}
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_point(Integer(2) + Integer(3)*I)
Point in UHP 3*I + 2
```

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```
>>> HyperbolicPlane().PD().get_point(Integer(0))
Point in PD 0

>>> HyperbolicPlane().KM().get_point((Integer(0),Integer(0)))
Point in KM (0, 0)

>>> HyperbolicPlane().HM().get_point((Integer(0),Integer(0),Integer(1)))
Point in HM (0, 0, 1)

>>> p = HyperbolicPlane().UHP().get_point(I, color='red')
>>> p.graphics_options()
{'color': 'red'}
```

```
sage: HyperbolicPlane().UHP().get_point(12)
Boundary point in UHP 12

sage: HyperbolicPlane().UHP().get_point(infinity)
Boundary point in UHP +Infinity

sage: HyperbolicPlane().PD().get_point(I)
Boundary point in PD I

sage: HyperbolicPlane().KM().get_point((0,-1))
Boundary point in KM (0, -1)
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_point(Integer(12))
Boundary point in UHP 12

>>> HyperbolicPlane().UHP().get_point(infinity)
Boundary point in UHP +Infinity

>>> HyperbolicPlane().PD().get_point(I)
Boundary point in PD I

>>> HyperbolicPlane().KM().get_point((Integer(0),-Integer(1)))
Boundary point in KM (0, -1)
```

is_bounded()

Return `True` if `self` is a bounded model.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().is_bounded()
True
sage: HyperbolicPlane().PD().is_bounded()
True
sage: HyperbolicPlane().KM().is_bounded()
True
sage: HyperbolicPlane().HM().is_bounded()
False
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().is_bounded()
True
>>> HyperbolicPlane().PD().is_bounded()
True
>>> HyperbolicPlane().KM().is_bounded()
True
>>> HyperbolicPlane().HM().is_bounded()
False
```

is_conformal()

Return `True` if `self` is a conformal model.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.is_conformal()
True
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.is_conformal()
True
```

is_isometry_group_projective()

Return `True` if the isometry group of `self` is projective.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.is_isometry_group_projective()
True
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.is_isometry_group_projective()
True
```

isometry_from_fixed_points(*repel*, *attract*)

Given two fixed points `repel` and `attract` as hyperbolic points return a hyperbolic isometry with `repel` as repelling fixed point and `attract` as attracting fixed point.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: PD = HyperbolicPlane().PD()
sage: PD.isometry_from_fixed_points(-i, i)
Isometry in PD
[   3/4  1/4*I]
[-1/4*I    3/4]
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
```

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```
>>> PD = HyperbolicPlane().PD()
>>> PD.isometry_from_fixed_points(-i, i)
Isometry in PD
[   3/4  1/4*I]
[-1/4*I    3/4]
```

```
sage: p, q = PD.get_point(1/2 + I/2), PD.get_point(6/13 + 9/13*I)
sage: PD.isometry_from_fixed_points(p, q)
Traceback (most recent call last):
...
ValueError: fixed points of hyperbolic elements must be ideal

sage: p, q = PD.get_point(4/5 + 3/5*I), PD.get_point(-I)
sage: PD.isometry_from_fixed_points(p, q)
Isometry in PD
[ 1/6*I - 2/3 -1/3*I - 1/6]
[ 1/3*I - 1/6 -1/6*I - 2/3]
```

```
>>> from sage.all import *
>>> p, q = PD.get_point(Integer(1)/Integer(2) + I/Integer(2)), PD.get_
↪point(Integer(6)/Integer(13) + Integer(9)/Integer(13)*I)
>>> PD.isometry_from_fixed_points(p, q)
Traceback (most recent call last):
...
ValueError: fixed points of hyperbolic elements must be ideal

>>> p, q = PD.get_point(Integer(4)/Integer(5) + Integer(3)/Integer(5)*I), PD.
↪get_point(-I)
>>> PD.isometry_from_fixed_points(p, q)
Isometry in PD
[ 1/6*I - 2/3 -1/3*I - 1/6]
[ 1/3*I - 1/6 -1/6*I - 2/3]
```

isometry_in_model(*A*)

Return `True` if the input matrix represents an isometry of the given model and `False` otherwise.

INPUT:

- `A` – a matrix that represents an isometry in the appropriate model

OUTPUT: boolean

EXAMPLES:

```
sage: HyperbolicPlane().UHP().isometry_in_model(identity_matrix(2))
True

sage: HyperbolicPlane().UHP().isometry_in_model(identity_matrix(3))
False
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().isometry_in_model(identity_matrix(Integer(2)))
True
```

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```
>>> HyperbolicPlane().UHP().isometry_in_model(identity_matrix(Integer(3)))
False
```

isometry_test(*A*)

Test whether an isometry `A` is in the model.

If the isometry is in the model, do nothing. Otherwise, raise a `ValueError`.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().isometry_test(identity_matrix(2))
sage: HyperbolicPlane().UHP().isometry_test(matrix(2, [I,1,2,1]))
Traceback (most recent call last):
...
ValueError:
[I 1]
[2 1] is not a valid isometry in the UHP model
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().isometry_test(identity_matrix(Integer(2)))
>>> HyperbolicPlane().UHP().isometry_test(matrix(Integer(2), [I,Integer(1),
↪Integer(2),Integer(1)]))
Traceback (most recent call last):
...
ValueError:
[I 1]
[2 1] is not a valid isometry in the UHP model
```

name()

Return the name of this model.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.name()
'Upper Half Plane Model'
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.name()
'Upper Half Plane Model'
```

point_in_model(*p*)

Return `True` if the point `p` is in the interior of the given model and `False` otherwise.

INPUT:

- any object that can converted into a complex number

OUTPUT: boolean

EXAMPLES:

```
sage: HyperbolicPlane().UHP().point_in_model(I)
True
sage: HyperbolicPlane().UHP().point_in_model(-I)
False
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().point_in_model(I)
True
>>> HyperbolicPlane().UHP().point_in_model(-I)
False
```

point_test(*p*)

Test whether a point is in the model. If the point is in the model, do nothing. Otherwise, raise a `ValueError`.

EXAMPLES:

```
sage: from sage.geometry.hyperbolic_space.hyperbolic_model import␣
↪HyperbolicModelUHP
sage: HyperbolicPlane().UHP().point_test(2 + I)
sage: HyperbolicPlane().UHP().point_test(2 - I)
Traceback (most recent call last):
...
ValueError: -I + 2 is not a valid point in the UHP model
```

```
>>> from sage.all import *
>>> from sage.geometry.hyperbolic_space.hyperbolic_model import␣
↪HyperbolicModelUHP
>>> HyperbolicPlane().UHP().point_test(Integer(2) + I)
>>> HyperbolicPlane().UHP().point_test(Integer(2) - I)
Traceback (most recent call last):
...
ValueError: -I + 2 is not a valid point in the UHP model
```

random_element(***kwargs*)

Return a random point in `self`.

The points are uniformly distributed over the rectangle $[-10, 10] \times [0, 10i]$ in the upper half plane model.

EXAMPLES:

```
sage: p = HyperbolicPlane().UHP().random_element()
sage: bool((p.coordinates().imag()) > 0)
True

sage: p = HyperbolicPlane().PD().random_element()
sage: HyperbolicPlane().PD().point_in_model(p.coordinates())
True

sage: p = HyperbolicPlane().KM().random_element()
sage: HyperbolicPlane().KM().point_in_model(p.coordinates())
True

sage: p = HyperbolicPlane().HM().random_element().coordinates()
```

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```
sage: bool((p[0]**2 + p[1]**2 - p[2]**2 - 1) < 10**-8)
True
```

```
>>> from sage.all import *
>>> p = HyperbolicPlane().UHP().random_element()
>>> bool((p.coordinates().imag()) > Integer(0))
True

>>> p = HyperbolicPlane().PD().random_element()
>>> HyperbolicPlane().PD().point_in_model(p.coordinates())
True

>>> p = HyperbolicPlane().KM().random_element()
>>> HyperbolicPlane().KM().point_in_model(p.coordinates())
True

>>> p = HyperbolicPlane().HM().random_element().coordinates()
>>> bool((p[Integer(0)]**Integer(2) + p[Integer(1)]**Integer(2) -␣
↪p[Integer(2)]**Integer(2) - Integer(1)) < Integer(10)**-Integer(8))
True
```

random_geodesic(***kwargs*)

Return a random hyperbolic geodesic.

Return the geodesic between two random points.

EXAMPLES:

```
sage: h = HyperbolicPlane().PD().random_geodesic()
sage: all( e.coordinates().abs() <= 1 for e in h.endpoints() )
True
```

```
>>> from sage.all import *
>>> h = HyperbolicPlane().PD().random_geodesic()
>>> all( e.coordinates().abs() <= Integer(1) for e in h.endpoints() )
True
```

random_isometry(*preserve_orientation=True*, ***kwargs*)

Return a random isometry in the model of `self`.

INPUT:

- `preserve_orientation` – if `True` return an orientation-preserving isometry

OUTPUT: a hyperbolic isometry

EXAMPLES:

```
sage: # needs scipy
sage: A = HyperbolicPlane().PD().random_isometry()
sage: A.preserves_orientation()
True
sage: B = HyperbolicPlane().PD().random_isometry(preserve_orientation=False)
sage: B.preserves_orientation()
False
```

```
>>> from sage.all import *
>>> # needs scipy
>>> A = HyperbolicPlane().PD().random_isometry()
>>> A.preserves_orientation()
True
>>> B = HyperbolicPlane().PD().random_isometry(preserve_orientation=False)
>>> B.preserves_orientation()
False
```

random_point(***kwargs*)

Return a random point of `self`.

The points are uniformly distributed over the rectangle $[-10, 10] \times [0, 10i]$ in the upper half plane model.

EXAMPLES:

```
sage: p = HyperbolicPlane().UHP().random_point()
sage: bool((p.coordinates().imag()) > 0)
True

sage: PD = HyperbolicPlane().PD()
sage: p = PD.random_point()
sage: PD.point_in_model(p.coordinates())
True
```

```
>>> from sage.all import *
>>> p = HyperbolicPlane().UHP().random_point()
>>> bool((p.coordinates().imag()) > Integer(0))
True

>>> PD = HyperbolicPlane().PD()
>>> p = PD.random_point()
>>> PD.point_in_model(p.coordinates())
True
```

short_name()

Return the short name of this model.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.short_name()
'UHP'
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.short_name()
'UHP'
```

class sage.geometry.hyperbolic_space.hyperbolic_model.**HyperbolicModelHM**(*space*)

Bases: *HyperbolicModel*

Hyperboloid Model.

boundary_point_in_model(*p*)

Return `False` since the Hyperboloid model has no boundary points.

EXAMPLES:

```
sage: HM = HyperbolicPlane().HM()
sage: HM.boundary_point_in_model((0,0,1))
False
sage: HM.boundary_point_in_model((1,0,sqrt(2)))
False
sage: HM.boundary_point_in_model((1,2,1))
False
```

```
>>> from sage.all import *
>>> HM = HyperbolicPlane().HM()
>>> HM.boundary_point_in_model((Integer(0),Integer(0),Integer(1)))
False
>>> HM.boundary_point_in_model((Integer(1),Integer(0),sqrt(Integer(2))))
False
>>> HM.boundary_point_in_model((Integer(1),Integer(2),Integer(1)))
False
```

get_background_graphic(***bdry_options*)

Return a graphic object that makes the model easier to visualize. For the hyperboloid model, the background object is the hyperboloid itself.

EXAMPLES:

```
sage: H = HyperbolicPlane().HM().get_background_graphic()                    #␣
↪needs sage.plot
```

```
>>> from sage.all import *
>>> H = HyperbolicPlane().HM().get_background_graphic()                    #␣
↪needs sage.plot
```

isometry_in_model(*A*)

Test that the matrix `A` is in the group $SO(2,1)^+$.

EXAMPLES:

```
sage: A = diagonal_matrix([1,1,-1])
sage: HyperbolicPlane().HM().isometry_in_model(A)
↪# needs scipy
True
```

```
>>> from sage.all import *
>>> A = diagonal_matrix([Integer(1),Integer(1),-Integer(1)])
>>> HyperbolicPlane().HM().isometry_in_model(A)                             #␣
↪needs scipy
True
```

point_in_model(*p*)

Check whether a complex number lies in the hyperboloid.

EXAMPLES:

```
sage: HM = HyperbolicPlane().HM()
sage: HM.point_in_model((0,0,1))
True
sage: HM.point_in_model((1,0,sqrt(2)))
True
sage: HM.point_in_model((1,2,1))
False
```

```
>>> from sage.all import *
>>> HM = HyperbolicPlane().HM()
>>> HM.point_in_model((Integer(0),Integer(0),Integer(1)))
True
>>> HM.point_in_model((Integer(1),Integer(0),sqrt(Integer(2))))
True
>>> HM.point_in_model((Integer(1),Integer(2),Integer(1)))
False
```

class sage.geometry.hyperbolic_space.hyperbolic_model.**HyperbolicModelKM**(*space*)

Bases: *HyperbolicModel*

Klein Model.

boundary_point_in_model(*p*)

Check whether a point lies in the unit circle, which corresponds to the ideal boundary of the hyperbolic plane in the Klein model.

EXAMPLES:

```
sage: KM = HyperbolicPlane().KM()
sage: KM.boundary_point_in_model((1, 0))
True
sage: KM.boundary_point_in_model((1/2, 1/2))
False
sage: KM.boundary_point_in_model((1, .2))
False
```

```
>>> from sage.all import *
>>> KM = HyperbolicPlane().KM()
>>> KM.boundary_point_in_model((Integer(1), Integer(0)))
True
>>> KM.boundary_point_in_model((Integer(1)/Integer(2), Integer(1)/Integer(2)))
False
>>> KM.boundary_point_in_model((Integer(1), RealNumber('.2')))
False
```

get_background_graphic(***bdry_options*)

Return a graphic object that makes the model easier to visualize.

For the Klein model, the background object is the ideal boundary.

EXAMPLES:

```
sage: circ = HyperbolicPlane().KM().get_background_graphic()                #␣
↪needs sage.plot
```

```
>>> from sage.all import *
>>> circ = HyperbolicPlane().KM().get_background_graphic()                    #␣
↪needs sage.plot
```

isometry_in_model(*A*)

Check if the given matrix `A` is in the group $SO(2,1)$.

EXAMPLES:

```
sage: A = matrix(3, [[1, 0, 0], [0, 17/8, 15/8], [0, 15/8, 17/8]])
sage: HyperbolicPlane().KM().isometry_in_model(A)                              #␣
↪needs scipy
True
```

```
>>> from sage.all import *
>>> A = matrix(Integer(3), [[Integer(1), Integer(0), Integer(0)], [Integer(0),
↪ Integer(17)/Integer(8), Integer(15)/Integer(8)], [Integer(0), Integer(15)/
↪Integer(8), Integer(17)/Integer(8)]])
>>> HyperbolicPlane().KM().isometry_in_model(A)                               #␣
↪needs scipy
True
```

point_in_model(*p*)

Check whether a point lies in the open unit disk.

EXAMPLES:

```
sage: KM = HyperbolicPlane().KM()
sage: KM.point_in_model((1, 0))
False
sage: KM.point_in_model((1/2, 1/2))
True
sage: KM.point_in_model((1, .2))
False
```

```
>>> from sage.all import *
>>> KM = HyperbolicPlane().KM()
>>> KM.point_in_model((Integer(1), Integer(0)))
False
>>> KM.point_in_model((Integer(1)/Integer(2), Integer(1)/Integer(2)))
True
>>> KM.point_in_model((Integer(1), RealNumber('.2')))
False
```

class sage.geometry.hyperbolic_space.hyperbolic_model.**HyperbolicModelPD**(*space*)

Bases: *HyperbolicModel*

Poincaré Disk Model.

boundary_point_in_model(*p*)

Check whether a complex number lies in the open unit disk.

EXAMPLES:

```
sage: PD = HyperbolicPlane().PD()
sage: PD.boundary_point_in_model(1.00)
True
sage: PD.boundary_point_in_model(1/2 + I/2)
False
sage: PD.boundary_point_in_model(1 + .2*I)
False
```

```
>>> from sage.all import *
>>> PD = HyperbolicPlane().PD()
>>> PD.boundary_point_in_model(RealNumber('1.00'))
True
>>> PD.boundary_point_in_model(Integer(1)/Integer(2) + I/Integer(2))
False
>>> PD.boundary_point_in_model(Integer(1) + RealNumber('.2')*I)
False
```

get_background_graphic(***bdry_options*)

Return a graphic object that makes the model easier to visualize.

For the Poincaré disk, the background object is the ideal boundary.

EXAMPLES:

```
sage: circ = HyperbolicPlane().PD().get_background_graphic()                #␣
↪needs sage.plot
```

```
>>> from sage.all import *
>>> circ = HyperbolicPlane().PD().get_background_graphic()                  #␣
↪needs sage.plot
```

isometry_in_model(*A*)

Check if the given matrix `A` is in the group $U(1,1)$.

EXAMPLES:

```
sage: z = [CC.random_element() for k in range(2)]; z.sort(key=abs)
sage: A = matrix(2,[z[1], z[0],z[0].conjugate(),z[1].conjugate()])
sage: HyperbolicPlane().PD().isometry_in_model(A)
True
```

```
>>> from sage.all import *
>>> z = [CC.random_element() for k in range(Integer(2))]; z.sort(key=abs)
>>> A = matrix(Integer(2),[z[Integer(1)], z[Integer(0)],z[Integer(0)].
↪conjugate(),z[Integer(1)].conjugate()])
>>> HyperbolicPlane().PD().isometry_in_model(A)
True
```

point_in_model(*p*)

Check whether a complex number lies in the open unit disk.

EXAMPLES:

```
sage: PD = HyperbolicPlane().PD()
sage: PD.point_in_model(1.00)
False
sage: PD.point_in_model(1/2 + I/2)
True
sage: PD.point_in_model(1 + .2*I)
False
```

```
>>> from sage.all import *
>>> PD = HyperbolicPlane().PD()
>>> PD.point_in_model(RealNumber('1.00'))
False
>>> PD.point_in_model(Integer(1)/Integer(2) + I/Integer(2))
True
>>> PD.point_in_model(Integer(1) + RealNumber('.2')*I)
False
```

class sage.geometry.hyperbolic_space.hyperbolic_model.**HyperbolicModelUHP**(*space*)

Bases: *HyperbolicModel*

Upper Half Plane model.

Element

alias of *HyperbolicPointUHP*

boundary_point_in_model(*p*)

Check whether a complex number is a real number or `\infty`. In the `UHP.model_name_name`, this is the ideal boundary of hyperbolic space.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.boundary_point_in_model(1 + I)
False
sage: UHP.boundary_point_in_model(infinity)
True
sage: UHP.boundary_point_in_model(CC(infinity))
True
sage: UHP.boundary_point_in_model(RR(infinity))
True
sage: UHP.boundary_point_in_model(1)
True
sage: UHP.boundary_point_in_model(12)
True
sage: UHP.boundary_point_in_model(1 - I)
False
sage: UHP.boundary_point_in_model(-2*I)
False
sage: UHP.boundary_point_in_model(0)
True
sage: UHP.boundary_point_in_model(I)
False
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.boundary_point_in_model(Integer(1) + I)
False
>>> UHP.boundary_point_in_model(infinity)
True
>>> UHP.boundary_point_in_model(CC(infinity))
True
>>> UHP.boundary_point_in_model(RR(infinity))
True
>>> UHP.boundary_point_in_model(Integer(1))
True
>>> UHP.boundary_point_in_model(Integer(12))
True
>>> UHP.boundary_point_in_model(Integer(1) - I)
False
>>> UHP.boundary_point_in_model(-Integer(2)*I)
False
>>> UHP.boundary_point_in_model(Integer(0))
True
>>> UHP.boundary_point_in_model(I)
False
```

get_background_graphic(***bdry_options*)

Return a graphic object that makes the model easier to visualize. For the upper half space, the background object is the ideal boundary.

EXAMPLES:

```
sage: hp = HyperbolicPlane().UHP().get_background_graphic()                    #␣
↪needs sage.plot
```

```
>>> from sage.all import *
>>> hp = HyperbolicPlane().UHP().get_background_graphic()                    #␣
↪needs sage.plot
```

isometry_from_fixed_points(*repel*, *attract*)

Given two fixed points `repel` and `attract` as complex numbers return a hyperbolic isometry with `repel` as repelling fixed point and `attract` as attracting fixed point.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.isometry_from_fixed_points(2 + I, 3 + I)
Traceback (most recent call last):
...
ValueError: fixed points of hyperbolic elements must be ideal

sage: UHP.isometry_from_fixed_points(2, 0)
Isometry in UHP
[  -1    0]
[-1/3 -1/3]
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.isometry_from_fixed_points(Integer(2) + I, Integer(3) + I)
Traceback (most recent call last):
...
ValueError: fixed points of hyperbolic elements must be ideal

>>> UHP.isometry_from_fixed_points(Integer(2), Integer(0))
Isometry in UHP
[  -1    0]
[-1/3 -1/3]
```

isometry_in_model(A)

Check that A acts as an isometry on the upper half plane. That is, A must be an invertible 2×2 matrix with real entries.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: A = matrix(2,[1,2,3,4])
sage: UHP.isometry_in_model(A)
True
sage: B = matrix(2,[I,2,4,1])
sage: UHP.isometry_in_model(B)
False
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> A = matrix(Integer(2),[Integer(1),Integer(2),Integer(3),Integer(4)])
>>> UHP.isometry_in_model(A)
True
>>> B = matrix(Integer(2),[I,Integer(2),Integer(4),Integer(1)])
>>> UHP.isometry_in_model(B)
False
```

An example of a matrix A such that $\det(A) \neq 1$, but the A acts isometrically:

```
sage: C = matrix(2,[10,0,0,10])
sage: UHP.isometry_in_model(C)
True
```

```
>>> from sage.all import *
>>> C = matrix(Integer(2),[Integer(10),Integer(0),Integer(0),Integer(10)])
>>> UHP.isometry_in_model(C)
True
```

point_in_model(p)

Check whether a complex number lies in the open upper half plane.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.point_in_model(1 + I)
```

(continues on next page)

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```
True
sage: UHP.point_in_model(infinity)
False
sage: UHP.point_in_model(CC(infinity))
False
sage: UHP.point_in_model(RR(infinity))
False
sage: UHP.point_in_model(1)
False
sage: UHP.point_in_model(12)
False
sage: UHP.point_in_model(1 - I)
False
sage: UHP.point_in_model(-2*I)
False
sage: UHP.point_in_model(I)
True
sage: UHP.point_in_model(0) # Not interior point
False
```

```
>>> from sage.all import *
>>> UHP = HyperbolicPlane().UHP()
>>> UHP.point_in_model(Integer(1) + I)
True
>>> UHP.point_in_model(infinity)
False
>>> UHP.point_in_model(CC(infinity))
False
>>> UHP.point_in_model(RR(infinity))
False
>>> UHP.point_in_model(Integer(1))
False
>>> UHP.point_in_model(Integer(12))
False
>>> UHP.point_in_model(Integer(1) - I)
False
>>> UHP.point_in_model(-Integer(2)*I)
False
>>> UHP.point_in_model(I)
True
>>> UHP.point_in_model(Integer(0)) # Not interior point
False
```

random_isometry(*preserve_orientation=True*, ***kwargs*)

Return a random isometry in the Upper Half Plane model.

INPUT:

- `preserve_orientation` – if `True` return an orientation-preserving isometry

OUTPUT: a hyperbolic isometry

EXAMPLES:

```
sage: A = HyperbolicPlane().UHP().random_isometry()                          #␣
↪needs scipy
sage: B = HyperbolicPlane().UHP().random_isometry(preserve_orientation=False)␣
↪           # needs scipy
sage: B.preserves_orientation()                                              #␣
↪needs scipy
False
```

```
>>> from sage.all import *
>>> A = HyperbolicPlane().UHP().random_isometry()                            #␣
↪needs scipy
>>> B = HyperbolicPlane().UHP().random_isometry(preserve_orientation=False)  ␣
↪         # needs scipy
>>> B.preserves_orientation()                                                #␣
↪needs scipy
False
```

random_point(***kwargs*)

Return a random point in the upper half plane. The points are uniformly distributed over the rectangle $[-10, 10] \times [0, 10i]$.

EXAMPLES:

```
sage: p = HyperbolicPlane().UHP().random_point().coordinates()
sage: bool((p.imag()) > 0)
True
```

```
>>> from sage.all import *
>>> p = HyperbolicPlane().UHP().random_point().coordinates()
>>> bool((p.imag()) > Integer(0))
True
```

CHAPTER

FIVE

INTERFACE TO HYPERBOLIC MODELS

This module provides a convenient interface for interacting with models of hyperbolic space as well as their points, geodesics, and isometries.

The primary point of this module is to allow the code that implements hyperbolic space to be sufficiently decoupled while still providing a convenient user experience.

The interfaces are by default given abbreviated names. For example, UHP (upper half plane model), PD (Poincaré disk model), KM (Klein disk model), and HM (hyperboloid model).

Note

All of the current models of 2 dimensional hyperbolic space use the upper half plane model for their computations. This can lead to some problems, such as long coordinate strings for symbolic points. For example, the vector `(1, 0, sqrt(2))` defines a point in the hyperboloid model. Performing mapping this point to the upper half plane and performing computations there may return with vector whose components are unsimplified strings have several `sqrt(2)`'s. Presently, this drawback is outweighted by the rapidity with which new models can be implemented.

AUTHORS:

- Greg Laun (2013): Initial version.
- Rania Amer, Jean-Philippe Burelle, Bill Goldman, Zach Groton, Jeremy Lent, Leila Vaden, Derrick Wigglesworth (2011): many of the methods spread across the files.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_point(2 + I)
Point in UHP I + 2

sage: HyperbolicPlane().PD().get_point(1/2 + I/2)
Point in PD 1/2*I + 1/2
```

```
>>> from sage.all import *
>>> HyperbolicPlane().UHP().get_point(Integer(2) + I)
Point in UHP I + 2

>>> HyperbolicPlane().PD().get_point(Integer(1)/Integer(2) + I/Integer(2))
Point in PD 1/2*I + 1/2
```

class sage.geometry.hyperbolic_space.hyperbolic_interface.**HyperbolicModels**(*base*)

Bases: Category_realization_of_parent

The category of hyperbolic models of hyperbolic space.

class ParentMethods

Bases: object

super_categories()

The super categories of self.

EXAMPLES:

```
sage: from sage.geometry.hyperbolic_space.hyperbolic_interface import␣
↪HyperbolicModels
sage: H = HyperbolicPlane()
sage: models = HyperbolicModels(H)
sage: models.super_categories()
[Category of metric spaces,
 Category of realizations of Hyperbolic plane]
```

```
>>> from sage.all import *
>>> from sage.geometry.hyperbolic_space.hyperbolic_interface import␣
↪HyperbolicModels
>>> H = HyperbolicPlane()
>>> models = HyperbolicModels(H)
>>> models.super_categories()
[Category of metric spaces,
 Category of realizations of Hyperbolic plane]
```

class sage.geometry.hyperbolic_space.hyperbolic_interface.**HyperbolicPlane**

Bases: Parent, UniqueRepresentation

The hyperbolic plane $\mathbb{H}^2$.

Here are the models currently implemented:

- UHP – upper half plane
- PD – Poincaré disk
- KM – Klein disk
- HM – hyperboloid model

HM

alias of *HyperbolicModelHM*

Hyperboloid

alias of *HyperbolicModelHM*

KM

alias of *HyperbolicModelKM*

KleinDisk

alias of *HyperbolicModelKM*

PD

alias of *HyperbolicModelPD*

PoincareDisk

alias of *HyperbolicModelPD*

UHP

alias of *HyperbolicModelUHP*

UpperHalfPlane

alias of *HyperbolicModelUHP*

a_realization()

Return a realization of `self`.

EXAMPLES:

```
sage: H = HyperbolicPlane()
sage: H.a_realization()
Hyperbolic plane in the Upper Half Plane Model
```

```
>>> from sage.all import *
>>> H = HyperbolicPlane()
>>> H.a_realization()
Hyperbolic plane in the Upper Half Plane Model
```

sage.geometry.hyperbolic_space.hyperbolic_interface.**HyperbolicSpace**(*n*)

Return `n` dimensional hyperbolic space.

EXAMPLES:

```
sage: from sage.geometry.hyperbolic_space.hyperbolic_interface import␣
↪HyperbolicSpace
sage: HyperbolicSpace(2)
Hyperbolic plane
```

```
>>> from sage.all import *
>>> from sage.geometry.hyperbolic_space.hyperbolic_interface import␣
↪HyperbolicSpace
>>> HyperbolicSpace(Integer(2))
Hyperbolic plane
```

CHAPTER

SIX

INDICES AND TABLES

- Index
- Module Index
- Search Page

PYTHON MODULE INDEX

g

INDEX

G

H

I

K

L

M

N

P

R

S

T

U