Diophantine approximation

Release 10.6

The Sage Development Team

Jun 27, 2025

CONTENTS

The diophantine approximation deals with the approximation of real numbers (or real vectors) with rational numbers (or rational vectors). See the article Wikipedia article Diophantine_approximation for more information.

CHAPTER

ONE

CONTINUED FRACTIONS

A continued fraction is a representation of a real number in terms of a sequence of integers denoted $[a_0; a_1, a_2, \ldots]$. The well known decimal expansion is another way of representing a real number by a sequence of integers. The value of a continued fraction is defined recursively as:

$$[a_0; a_1, a_2, \ldots] = a_0 + \frac{1}{[a_1; a_2, \ldots]} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ldots}}}$$

In this expansion, all coefficients a_n are integers and only the value a_0 may be non positive. Note that a_0 is nothing else but the floor (this remark provides a way to build the continued fraction expansion from a given real number). As examples

$$\frac{45}{38} = 1 + \frac{1}{5 + \frac{1}{2 + \frac{1}{3}}}$$

$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \frac{1}{\ldots}}}}}$$

It is quite remarkable that

- any real number admits a unique continued fraction expansion
- finite expansions correspond to rationals
- ultimately periodic expansions correspond to quadratic numbers (ie numbers of the form $a + b\sqrt{D}$ with a and b rationals and D square free positive integer)
- two real numbers x and y have the same tail (up to a shift) in their continued fraction expansion if and only if there are integers a, b, c, d with $|ad - bc| = 1$ and such that $y = (ax + b)/(cx + d)$.

Moreover, the rational numbers obtained by truncation of the expansion of a real number gives its so-called best approximations. For more informations on continued fractions, you may have a look at Wikipedia article Continued_fraction.

EXAMPLES:

If you want to create the continued fraction of some real number you may either use its method continued_fraction (if it exists) or call `continued_fraction()`:

```
sage: (13/27).continued_fraction()
[0; 2, 13]
sage: 0 + 1/(2 + 1/13)
13/27

sage: continued_fraction(22/45)
[0; 2, 22]
sage: 0 + 1/(2 + 1/22)
22/45

sage: continued_fraction(pi)                                                #␣
↪needs sage.symbolic
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: continued_fraction_list(pi, nterms=5)                                 #␣
↪needs sage.symbolic
[3, 7, 15, 1, 292]

sage: x = polygen(ZZ, 'x')
sage: K.<cbrt5> = NumberField(x^3 - 5, embedding=1.709)                     #␣
↪needs sage.rings.number_field
sage: continued_fraction(cbrt5)                                             #␣
↪needs sage.rings.number_field
[1; 1, 2, 2, 4, 3, 3, 1, 5, 1, 1, 4, 10, 17, 1, 14, 1, 1, 3052, 1, ...]
```

```
>>> from sage.all import *
>>> (Integer(13)/Integer(27)).continued_fraction()
[0; 2, 13]
>>> Integer(0) + Integer(1)/(Integer(2) + Integer(1)/Integer(13))
13/27

>>> continued_fraction(Integer(22)/Integer(45))
[0; 2, 22]
>>> Integer(0) + Integer(1)/(Integer(2) + Integer(1)/Integer(22))
22/45

>>> continued_fraction(pi)                                                #␣
↪needs sage.symbolic
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
>>> continued_fraction_list(pi, nterms=Integer(5))                          ␣
↪     # needs sage.symbolic
[3, 7, 15, 1, 292]

>>> x = polygen(ZZ, 'x')
>>> K = NumberField(x**Integer(3) - Integer(5), embedding=RealNumber('1.709'), names=(
↪'cbrt5',)); (cbrt5,) = K._first_ngens(1)# needs sage.rings.number_field
>>> continued_fraction(cbrt5)                                             #␣
↪needs sage.rings.number_field
[1; 1, 2, 2, 4, 3, 3, 1, 5, 1, 1, 4, 10, 17, 1, 14, 1, 1, 3052, 1, ...]
```

It is also possible to create a continued fraction from a list of partial quotients:

```
sage: continued_fraction([-3,1,2,3,4,1,2])
[-3; 1, 2, 3, 4, 1, 2]
```

```
>>> from sage.all import *
>>> continued_fraction([-Integer(3),Integer(1),Integer(2),Integer(3),Integer(4),
↪Integer(1),Integer(2)])
[-3; 1, 2, 3, 4, 1, 2]
```

Even infinite:

```
sage: w = words.ThueMorseWord([1,2]); w                                      #␣
↪needs sage.combinat
word: 1221211221121221211212211221211221121221...
sage: continued_fraction(w)                                                  #␣
↪needs sage.combinat
[1; 2, 2, 1, 2, 1, 1, 2, 2, 1...]
```

```
>>> from sage.all import *
>>> w = words.ThueMorseWord([Integer(1),Integer(2)]); w                       ␣
↪               # needs sage.combinat
word: 1221211221121221211212211221211221121221...
>>> continued_fraction(w)                                                   #␣
↪needs sage.combinat
[1; 2, 2, 1, 2, 1, 1, 2, 2, 1...]
```

To go back and forth between the value (as a real number) and the partial quotients (seen as a finite or infinite list) you can use the methods `quotients` and `value`:

```
sage: cf = (13/27).continued_fraction()
sage: cf.quotients()
[0, 2, 13]
sage: cf.value()
13/27

sage: cf = continued_fraction(pi)                                            #␣
↪needs sage.symbolic
sage: cf.quotients()                                                         #␣
↪needs sage.symbolic
lazy list [3, 7, 15, ...]
sage: cf.value()                                                             #␣
↪needs sage.symbolic
pi

sage: # needs sage.combinat
sage: w = words.FibonacciWord([1,2])
sage: cf = continued_fraction(w)
sage: cf.quotients()
word: 1211212112112121121211211212112112121121...
sage: v = cf.value(); v
1.387954587967143?
sage: v.n(digits=100)
1.387954587967142336919313859873185477878152452498532271894917289826418577622648932169
↪885237034242967
sage: v.continued_fraction()
[1; 2, 1, 1, 2, 1, 2, 1, 1, 2, 1, 1, 2, 1, 2, 1, 1, 2, 1, 2...]
```

```
>>> from sage.all import *
>>> cf = (Integer(13)/Integer(27)).continued_fraction()
>>> cf.quotients()
[0, 2, 13]
>>> cf.value()
13/27

>>> cf = continued_fraction(pi)                                          #␣
↪needs sage.symbolic
>>> cf.quotients()                                                       #␣
↪needs sage.symbolic
lazy list [3, 7, 15, ...]
>>> cf.value()                                                           #␣
↪needs sage.symbolic
pi

>>> # needs sage.combinat
>>> w = words.FibonacciWord([Integer(1),Integer(2)])
>>> cf = continued_fraction(w)
>>> cf.quotients()
word: 1211212112112121121211211212112112121121...
>>> v = cf.value(); v
1.387954587967143?
>>> v.n(digits=Integer(100))
1.387954587967142336919313859873185477878152452498532271894917289826418577622648932169
↪885237034242967
>>> v.continued_fraction()
[1; 2, 1, 1, 2, 1, 2, 1, 1, 2, 1, 1, 2, 1, 2, 1, 1, 2, 1, 2...]
```

Recall that quadratic numbers correspond to ultimately periodic continued fractions. For them special methods give access to preperiod and period:

```
sage: # needs sage.rings.number_field
sage: K.<sqrt2> = QuadraticField(2)
sage: cf = continued_fraction(sqrt2); cf
[1; (2)*]
sage: cf.value()
sqrt2
sage: cf.preperiod()
(1,)
sage: cf.period()
(2,)

sage: cf = (3*sqrt2 + 1/2).continued_fraction(); cf                        #␣
↪needs sage.rings.number_field
[4; (1, 2, 1, 7)*]

sage: cf = continued_fraction([(1,2,3),(1,4)]); cf
[1; 2, 3, (1, 4)*]
sage: cf.value()                                                           #␣
↪needs sage.rings.number_field
-2/23*sqrt2 + 36/23
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = QuadraticField(Integer(2), names=('sqrt2',)); (sqrt2,) = K._first_ngens(1)
>>> cf = continued_fraction(sqrt2); cf
[1; (2)*]
>>> cf.value()
sqrt2
>>> cf.preperiod()
(1,)
>>> cf.period()
(2,)

>>> cf = (Integer(3)*sqrt2 + Integer(1)/Integer(2)).continued_fraction(); cf
↪                         # needs sage.rings.number_field
[4; (1, 2, 1, 7)*]

>>> cf = continued_fraction([(Integer(1),Integer(2),Integer(3)),(Integer(1),
↪Integer(4))]); cf
[1; 2, 3, (1, 4)*]
>>> cf.value()                                                              #
↪needs sage.rings.number_field
-2/23*sqrt2 + 36/23
```

On the following we can remark how the tail may change even in the same quadratic field:

```
sage: for i in range(20): print(continued_fraction(i*sqrt2))                 #
↪needs sage.rings.number_field
[0]
[1; (2)*]
[2; (1, 4)*]
[4; (4, 8)*]
[5; (1, 1, 1, 10)*]
[7; (14)*]
...
[24; (24, 48)*]
[25; (2, 5, 6, 5, 2, 50)*]
[26; (1, 6, 1, 2, 3, 2, 26, 2, 3, 2, 1, 6, 1, 52)*]
```

```
>>> from sage.all import *
>>> for i in range(Integer(20)): print(continued_fraction(i*sqrt2))
↪     # needs sage.rings.number_field
[0]
[1; (2)*]
[2; (1, 4)*]
[4; (4, 8)*]
[5; (1, 1, 1, 10)*]
[7; (14)*]
...
[24; (24, 48)*]
[25; (2, 5, 6, 5, 2, 50)*]
[26; (1, 6, 1, 2, 3, 2, 26, 2, 3, 2, 1, 6, 1, 52)*]
```

Nevertheless, the tail is preserved under invertible integer homographies:

```
sage: # needs sage.modular sage.rings.number_field
sage: apply_homography =  lambda m,z: (m[0,0]*z + m[0,1]) / (m[1,0]*z + m[1,1])
sage: m1 = SL2Z([60,13,83,18])
sage: m2 = SL2Z([27,80,28,83])
sage: a = sqrt2/3
sage: a.continued_fraction()
[0; 2, (8, 4)*]
sage: b = apply_homography(m1, a)
sage: b.continued_fraction()
[0; 1, 2, 1, 1, 1, 1, 6, (8, 4)*]
sage: c = apply_homography(m2, a)
sage: c.continued_fraction()
[0; 1, 26, 1, 2, 2, (8, 4)*]
sage: d = apply_homography(m1**2*m2**3, a)
sage: d.continued_fraction()
[0; 1, 2, 1, 1, 1, 1, 5, 2, 1, 1, 1, 1, 5, 26, 1, 2, 1, 26, 1, 2, 1, 26, 1, 2, 2, (8,␣
↪4)*]
```

```
>>> from sage.all import *
>>> # needs sage.modular sage.rings.number_field
>>> apply_homography =  lambda m,z: (m[Integer(0),Integer(0)]*z + m[Integer(0),
↪Integer(1)]) / (m[Integer(1),Integer(0)]*z + m[Integer(1),Integer(1)])
>>> m1 = SL2Z([Integer(60),Integer(13),Integer(83),Integer(18)])
>>> m2 = SL2Z([Integer(27),Integer(80),Integer(28),Integer(83)])
>>> a = sqrt2/Integer(3)
>>> a.continued_fraction()
[0; 2, (8, 4)*]
>>> b = apply_homography(m1, a)
>>> b.continued_fraction()
[0; 1, 2, 1, 1, 1, 1, 6, (8, 4)*]
>>> c = apply_homography(m2, a)
>>> c.continued_fraction()
[0; 1, 26, 1, 2, 2, (8, 4)*]
>>> d = apply_homography(m1**Integer(2)*m2**Integer(3), a)
>>> d.continued_fraction()
[0; 1, 2, 1, 1, 1, 1, 5, 2, 1, 1, 1, 1, 5, 26, 1, 2, 1, 26, 1, 2, 1, 26, 1, 2, 2, (8,␣
↪4)*]
```

Todo

- Improve numerical approximation (the method `_mpfr_()` is quite slow compared to the same method for an element of a number field)
- Make a class for generalized continued fractions of the form $a_0 + b_0/(a_1 + b_1/(...))$ (the standard continued fractions are when all $b_n = 1$ while the Hirzebruch-Jung continued fractions are the one for which $b_n = -1$ for all n). See Wikipedia article Generalized_continued_fraction.
- look at the function ContinuedFractionApproximationOfRoot in GAP

AUTHORS:

- Vincent Delecroix (2014): cleaning, refactorisation, documentation from the old implementation in `contfrac` (Issue #14567).

class sage.rings.continued_fraction.**ContinuedFraction_base**

Bases: SageObject

Base class for (standard) continued fractions.

If you want to implement your own continued fraction, simply derived from this class and implement the following methods:

- `def quotient(self, n)`: return the `n`-th quotient of `self` as a Sage integer
- `def length(self)`: the number of partial quotients of `self` as a Sage integer or `Infinity`.

and optionally:

- `def value(self)`: return the value of `self` (an exact real number)

This base class will provide:

- computation of convergents in *convergent()*, *numerator()* and *denominator()*
- comparison with other continued fractions (see `__richcmp__()`)
- elementary arithmetic function *floor()*, *ceil()*, *sign()*
- accurate numerical approximations `_mpfr_()`

All other methods, in particular the ones involving binary operations like sum or product, rely on the optional method `value()` (and not on convergents) and may fail at execution if it is not implemented.

additive_order()

Return the additive order of this continued fraction, which we defined to be the additive order of its value.

EXAMPLES:

```
sage: continued_fraction(-1).additive_order()
+Infinity
sage: continued_fraction(0).additive_order()
1
```

```
>>> from sage.all import *
>>> continued_fraction(-Integer(1)).additive_order()
+Infinity
>>> continued_fraction(Integer(0)).additive_order()
1
```

apply_homography(*a*, *b*, *c*, *d*, *forward_value=False*)

Return the continued fraction of $(ax + b)/(cx + d)$.

This is computed using Gosper's algorithm, see `continued_fraction_gosper`.

INPUT:

- `a`, `b`, `c`, `d` – integers
- `forward_value` – boolean (default: `False`); whether the returned continued fraction is given the symbolic value of $(ax + b)/(cx + d)$ and not only the list of partial quotients obtained from Gosper's algorithm

EXAMPLES:

```
sage: (5 * 13/6 - 2) / (3 * 13/6 - 4)
53/15
sage: continued_fraction(13/6).apply_homography(5, -2,  3, -4).value()
53/15
```

```
>>> from sage.all import *
>>> (Integer(5) * Integer(13)/Integer(6) - Integer(2)) / (Integer(3) *␣
↪Integer(13)/Integer(6) - Integer(4))
53/15
>>> continued_fraction(Integer(13)/Integer(6)).apply_homography(Integer(5), -
↪Integer(2),  Integer(3), -Integer(4)).value()
53/15
```

We demonstrate now the effect of the optional argument `forward_value`:

```
sage: cf = continued_fraction(pi)                                          #␣
↪needs sage.symbolic
sage: h1 = cf.apply_homography(35, -27, 12, -5); h1                        #␣
↪needs sage.symbolic
[2; 1, 1, 6, 3, 1, 2, 1, 5, 3, 1, 1, 1, 1, 9, 12, 1, 1, 1, 3...]
sage: h1.value()                                                           #␣
↪needs sage.symbolic
2.536941776086946?

sage: h2 = cf.apply_homography(35, -27, 12, -5, forward_value=True); h2    #␣
↪needs sage.symbolic
[2; 1, 1, 6, 3, 1, 2, 1, 5, 3, 1, 1, 1, 1, 9, 12, 1, 1, 1, 3...]
sage: h2.value()                                                           #␣
↪needs sage.symbolic
(35*pi - 27)/(12*pi - 5)
```

```
>>> from sage.all import *
>>> cf = continued_fraction(pi)                                          #␣
↪needs sage.symbolic
>>> h1 = cf.apply_homography(Integer(35), -Integer(27), Integer(12), -
↪Integer(5)); h1                            # needs sage.symbolic
[2; 1, 1, 6, 3, 1, 2, 1, 5, 3, 1, 1, 1, 1, 9, 12, 1, 1, 1, 3...]
>>> h1.value()                                                           #␣
↪needs sage.symbolic
2.536941776086946?

>>> h2 = cf.apply_homography(Integer(35), -Integer(27), Integer(12), -
↪Integer(5), forward_value=True); h2      # needs sage.symbolic
[2; 1, 1, 6, 3, 1, 2, 1, 5, 3, 1, 1, 1, 1, 9, 12, 1, 1, 1, 3...]
>>> h2.value()                                                           #␣
↪needs sage.symbolic
(35*pi - 27)/(12*pi - 5)
```

REFERENCES:

- [Gos1972]
- [Knu1998] Exercise 4.5.3.15
- [LS1998]

ceil()

Return the ceil of `self`.

EXAMPLES:

```
sage: cf = continued_fraction([2,1,3,4])
sage: cf.ceil()
3
```

```
>>> from sage.all import *
>>> cf = continued_fraction([Integer(2),Integer(1),Integer(3),Integer(4)])
>>> cf.ceil()
3
```

convergent(*n*)

Return the `n`-th partial convergent to `self`.

EXAMPLES:

```
sage: a = continued_fraction(pi); a                                     #␣
↪needs sage.symbolic
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: a.convergent(3)                                                    #␣
↪needs sage.symbolic
355/113
sage: a.convergent(15)                                                   #␣
↪needs sage.symbolic
411557987/131002976
```

```
>>> from sage.all import *
>>> a = continued_fraction(pi); a                                       #␣
↪needs sage.symbolic
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
>>> a.convergent(Integer(3))                                              ␣
↪     # needs sage.symbolic
355/113
>>> a.convergent(Integer(15))                                             ␣
↪     # needs sage.symbolic
411557987/131002976
```

convergents()

Return the list of partial convergents of `self`.

If `self` is an infinite continued fraction, then the object returned is a `lazy_list_generic` which behave like an infinite list.

EXAMPLES:

```
sage: a = continued_fraction(23/157); a
[0; 6, 1, 4, 1, 3]
sage: a.convergents()
[0, 1/6, 1/7, 5/34, 6/41, 23/157]
```

```
>>> from sage.all import *
>>> a = continued_fraction(Integer(23)/Integer(157)); a
[0; 6, 1, 4, 1, 3]
>>> a.convergents()
[0, 1/6, 1/7, 5/34, 6/41, 23/157]
```

Todo

Add an example with infinite list.

denominator(*n*)

Return the denominator of the `n`-th partial convergent of `self`.

EXAMPLES:

```
sage: # needs sage.symbolic
sage: c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: c.denominator(0)
1
sage: c.denominator(12)
25510582
sage: c.denominator(152)
1255341492699841451528811722575401081588363886480089431843026103930863337221076748
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
>>> c.denominator(Integer(0))
1
>>> c.denominator(Integer(12))
25510582
>>> c.denominator(Integer(152))
1255341492699841451528811722575401081588363886480089431843026103930863337221076748
```

floor()

Return the floor of `self`.

EXAMPLES:

```
sage: cf = continued_fraction([2,1,2,3])
sage: cf.floor()
2
```

```
>>> from sage.all import *
>>> cf = continued_fraction([Integer(2),Integer(1),Integer(2),Integer(3)])
>>> cf.floor()
2
```

is_minus_one()

Test whether `self` is minus one.

EXAMPLES:

```
sage: continued_fraction(-1).is_minus_one()
True
sage: continued_fraction(1).is_minus_one()
False
sage: continued_fraction(0).is_minus_one()
False
sage: continued_fraction(-2).is_minus_one()
False
sage: continued_fraction([-1,1]).is_minus_one()
False
```

```
>>> from sage.all import *
>>> continued_fraction(-Integer(1)).is_minus_one()
True
>>> continued_fraction(Integer(1)).is_minus_one()
False
>>> continued_fraction(Integer(0)).is_minus_one()
False
>>> continued_fraction(-Integer(2)).is_minus_one()
False
>>> continued_fraction([-Integer(1),Integer(1)]).is_minus_one()
False
```

is_one()

Test whether `self` is one.

EXAMPLES:

```
sage: continued_fraction(1).is_one()
True
sage: continued_fraction(5/4).is_one()
False
sage: continued_fraction(0).is_one()
False
sage: continued_fraction(pi).is_one()                                  #␣
↪needs sage.symbolic
False
```

```
>>> from sage.all import *
>>> continued_fraction(Integer(1)).is_one()
True
>>> continued_fraction(Integer(5)/Integer(4)).is_one()
False
>>> continued_fraction(Integer(0)).is_one()
False
>>> continued_fraction(pi).is_one()                                  #␣
↪needs sage.symbolic
False
```

is_zero()

Test whether `self` is zero.

EXAMPLES:

```
sage: continued_fraction(0).is_zero()
True
sage: continued_fraction((0,1)).is_zero()
False
sage: continued_fraction(-1/2).is_zero()
False
sage: continued_fraction(pi).is_zero()                                          #␣
↪needs sage.symbolic
False
```

```
>>> from sage.all import *
>>> continued_fraction(Integer(0)).is_zero()
True
>>> continued_fraction((Integer(0),Integer(1))).is_zero()
False
>>> continued_fraction(-Integer(1)/Integer(2)).is_zero()
False
>>> continued_fraction(pi).is_zero()                                            #␣
↪needs sage.symbolic
False
```

multiplicative_order()

Return the multiplicative order of this continued fraction, which we defined to be the multiplicative order of its value.

EXAMPLES:

```
sage: continued_fraction(-1).multiplicative_order()
2
sage: continued_fraction(1).multiplicative_order()
1
sage: continued_fraction(pi).multiplicative_order()                             #␣
↪needs sage.symbolic
+Infinity
```

```
>>> from sage.all import *
>>> continued_fraction(-Integer(1)).multiplicative_order()
2
>>> continued_fraction(Integer(1)).multiplicative_order()
1
>>> continued_fraction(pi).multiplicative_order()                               #␣
↪needs sage.symbolic
+Infinity
```

n(*prec=None*, *digits=None*, *algorithm=None*)

Return a numerical approximation of this continued fraction with `prec` bits (or decimal `digits`) of precision.

INPUT:

- `prec` – precision in bits

- `digits` – precision in decimal digits (only used if `prec` is not given)
- `algorithm` – ignored for continued fractions

If neither `prec` nor `digits` is given, the default precision is 53 bits (roughly 16 digits).

EXAMPLES:

```
sage: w = words.FibonacciWord([1,3])                                    #␣
↪needs sage.combinat
sage: cf = continued_fraction(w); cf                                    #␣
↪needs sage.combinat
[1; 3, 1, 1, 3, 1, 3, 1, 1, 3, 1, 1, 3, 1, 3, 1, 1, 3, 1, 3...]
sage: cf.numerical_approx(prec=53)                                      #␣
↪needs sage.combinat
1.28102513329557
```

```
>>> from sage.all import *
>>> w = words.FibonacciWord([Integer(1),Integer(3)])                     ␣
↪               # needs sage.combinat
>>> cf = continued_fraction(w); cf                                     #␣
↪needs sage.combinat
[1; 3, 1, 1, 3, 1, 3, 1, 1, 3, 1, 1, 3, 1, 3, 1, 1, 3, 1, 3...]
>>> cf.numerical_approx(prec=Integer(53))                                ␣
↪      # needs sage.combinat
1.28102513329557
```

The method n is a shortcut to this one:

```
sage: cf.n(digits=25)                                                   #␣
↪needs sage.combinat
1.281025133295569815552930
sage: cf.n(digits=33)                                                   #␣
↪needs sage.combinat
1.28102513329556981555293038097590
```

```
>>> from sage.all import *
>>> cf.n(digits=Integer(25))                                             ␣
↪      # needs sage.combinat
1.281025133295569815552930
>>> cf.n(digits=Integer(33))                                             ␣
↪      # needs sage.combinat
1.28102513329556981555293038097590
```

numerator(n)

Return the numerator of the n-th partial convergent of `self`.

EXAMPLES:

```
sage: # needs sage.symbolic
sage: c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: c.numerator(0)
3
sage: c.numerator(12)
```

(continues on next page)

(continued from previous page)

```
80143857
sage: c.numerator(152)
394377161121226696274373881260074821315726659658874495172739349744692124535300
↪5283
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
>>> c.numerator(Integer(0))
3
>>> c.numerator(Integer(12))
80143857
>>> c.numerator(Integer(152))
394377161121226696274373881260074821315726659658874495172739349744692124535300
↪5283
```

numerical_approx(*prec=None*, *digits=None*, *algorithm=None*)

Return a numerical approximation of this continued fraction with `prec` bits (or decimal `digits`) of precision.

INPUT:

- `prec` – precision in bits
- `digits` – precision in decimal digits (only used if `prec` is not given)
- `algorithm` – ignored for continued fractions

If neither `prec` nor `digits` is given, the default precision is 53 bits (roughly 16 digits).

EXAMPLES:

```
sage: w = words.FibonacciWord([1,3])                                          #␣
↪needs sage.combinat
sage: cf = continued_fraction(w); cf                                          #␣
↪needs sage.combinat
[1; 3, 1, 1, 3, 1, 3, 1, 1, 3, 1, 1, 3, 1, 3, 1, 1, 3, 1, 3...]
sage: cf.numerical_approx(prec=53)                                            #␣
↪needs sage.combinat
1.28102513329557
```

```
>>> from sage.all import *
>>> w = words.FibonacciWord([Integer(1),Integer(3)])                           ␣
↪               # needs sage.combinat
>>> cf = continued_fraction(w); cf                                           #␣
↪needs sage.combinat
[1; 3, 1, 1, 3, 1, 3, 1, 1, 3, 1, 1, 3, 1, 3, 1, 1, 3, 1, 3...]
>>> cf.numerical_approx(prec=Integer(53))                                      ␣
↪      # needs sage.combinat
1.28102513329557
```

The method n is a shortcut to this one:

```
sage: cf.n(digits=25)                                                         #␣
↪needs sage.combinat
```

(continues on next page)

(continued from previous page)

```
1.281025133295569815552930
sage: cf.n(digits=33)                                                              #␣
↪needs sage.combinat
1.28102513329556981555293038097590
```

```
>>> from sage.all import *
>>> cf.n(digits=Integer(25))                                                        ␣
↪      # needs sage.combinat
1.281025133295569815552930
>>> cf.n(digits=Integer(33))                                                        ␣
↪      # needs sage.combinat
1.28102513329556981555293038097590
```

p(*n*)

Return the numerator of the n-th partial convergent of `self`.

EXAMPLES:

```
sage: # needs sage.symbolic
sage: c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: c.numerator(0)
3
sage: c.numerator(12)
80143857
sage: c.numerator(152)
3943771611212266962743738812600748213157266596588744951727393497446921245353005
↪5283
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
>>> c.numerator(Integer(0))
3
>>> c.numerator(Integer(12))
80143857
>>> c.numerator(Integer(152))
3943771611212266962743738812600748213157266596588744951727393497446921245353005
↪5283
```

q(*n*)

Return the denominator of the `n`-th partial convergent of `self`.

EXAMPLES:

```
sage: # needs sage.symbolic
sage: c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: c.denominator(0)
1
sage: c.denominator(12)
```

(continues on next page)

(continued from previous page)

```
25510582
sage: c.denominator(152)
125534149269984145152881172257540108158836388648008943184302610393086333722107
↪6748
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
>>> c.denominator(Integer(0))
1
>>> c.denominator(Integer(12))
25510582
>>> c.denominator(Integer(152))
125534149269984145152881172257540108158836388648008943184302610393086333722107
↪6748
```

quotients()

Return the list of partial quotients of `self`.

If `self` is an infinite continued fraction, then the object returned is a `lazy_list_generic` which behaves like an infinite list.

EXAMPLES:

```
sage: a = continued_fraction(23/157); a
[0; 6, 1, 4, 1, 3]
sage: a.quotients()
[0, 6, 1, 4, 1, 3]
```

```
>>> from sage.all import *
>>> a = continued_fraction(Integer(23)/Integer(157)); a
[0; 6, 1, 4, 1, 3]
>>> a.quotients()
[0, 6, 1, 4, 1, 3]
```

Todo

Add an example with infinite list.

sign()

Return the sign of `self` as an Integer.

The sign is defined to be `0` if `self` is `0`, `1` if `self` is positive and `-1` if `self` is negative.

EXAMPLES:

```
sage: continued_fraction(tan(pi/7)).sign()                                    #␣
↪needs sage.symbolic
1
sage: continued_fraction(-34/2115).sign()
-1
```

(continues on next page)

(continued from previous page)

```
sage: continued_fraction([0]).sign()
0
```

```
>>> from sage.all import *
>>> continued_fraction(tan(pi/Integer(7))).sign()                              ␣
↪     # needs sage.symbolic
1
>>> continued_fraction(-Integer(34)/Integer(2115)).sign()
-1
>>> continued_fraction([Integer(0)]).sign()
0
```

str(*nterms=10*, *unicode=False*, *join=True*)

Return a string representing this continued fraction.

INPUT:

- `nterms` – the maximum number of terms to use
- `unicode` – (default: `False`) whether to use unicode character
- `join` – (default: `True`) if `False` instead of returning a string return a list of string, each of them representing a line

EXAMPLES:

```
sage: print(continued_fraction(pi).str())                                       #␣
↪needs sage.symbolic
                             1
3 + ----------------------------------------------------
                                1
     7 + -----------------------------------------------
                                   1
          15 + -----------------------------------------
                                     1
                1 + ------------------------------------
                                         1
                     292 + -----------------------------
                                           1
                           1 + ------------------------
                                              1
                                1 + -------------------
                                                1
                                     1 + --------------
                                                   1
                                          2 + ---------
                                                1 + ...
sage: print(continued_fraction(pi).str(nterms=1))                               #␣
↪needs sage.symbolic
3 + ...
sage: print(continued_fraction(pi).str(nterms=2))                               #␣
↪needs sage.symbolic
        1
3 + ---------
```

(continues on next page)

(continued from previous page)

```
     7 + ...

sage: print(continued_fraction(243/354).str())
           1
-----------------------
             1
 1 + ------------------
                1
      2 + -------------
                  1
           5 + --------
                     1
                3 + ---
                     2
sage: continued_fraction(243/354).str(join=False)
['           1           ',
 '-----------------------',
 '             1         ',
 ' 1 + ------------------',
 '                1      ',
 '      2 + -------------',
 '                  1    ',
 '           5 + --------',
 '                     1 ',
 '                3 + ---',
 '                     2 ']

sage: print(continued_fraction(243/354).str(unicode=True))
           1
───────────────────────
             1
 1 + ──────────────────
                1
      2 + ─────────────
                  1
           5 + ────────
                     1
                3 + ───
                     2
```

```
>>> from sage.all import *
>>> print(continued_fraction(pi).str())                                 #␣
↪needs sage.symbolic
                             1
3 + ----------------------------------------------------
                                1
     7 + -----------------------------------------------
                                   1
          15 + -----------------------------------------
                                     1
                1 + ------------------------------------
                                         1
```

(continues on next page)

(continued from previous page)

```
                     292 + -----------------------------
                                           1
                            1 + ------------------------
                                              1
                                 1 + -------------------
                                                1
                                      1 + --------------
                                                   1
                                           2 + ---------
                                                1 + ...
>>> print(continued_fraction(pi).str(nterms=Integer(1)))
↪      # needs sage.symbolic
3 + ...
>>> print(continued_fraction(pi).str(nterms=Integer(2)))
↪      # needs sage.symbolic
        1
3 + ---------
     7 + ...

>>> print(continued_fraction(Integer(243)/Integer(354)).str())
           1
-----------------------
             1
 1 + ------------------
                1
      2 + -------------
                  1
           5 + --------
                     1
                3 + ---
                     2
>>> continued_fraction(Integer(243)/Integer(354)).str(join=False)
['           1           ',
 '-----------------------',
 '             1         ',
 ' 1 + ------------------',
 '                1      ',
 '      2 + -------------',
 '                  1    ',
 '           5 + --------',
 '                     1 ',
 '                3 + ---',
 '                     2 ']

>>> print(continued_fraction(Integer(243)/Integer(354)).str(unicode=True))
           1
───────────────────────
             1
 1 + ──────────────────
                1
      2 + ─────────────
                  1
```

(continues on next page)

(continued from previous page)

```
              5 + ────────
                        1
                   3 + ───
                        2
```

class sage.rings.continued_fraction.**ContinuedFraction_infinite**(*w*, *value=None*, *check=True*)

Bases: *ContinuedFraction_base*

A continued fraction defined by an infinite sequence of partial quotients.

EXAMPLES:

```
sage: t = continued_fraction(words.ThueMorseWord([1,2])); t                    #␣
↪needs sage.combinat
[1; 2, 2, 1, 2, 1, 1, 2, 2, 1...]
sage: t.n(digits=100)                                                          #␣
↪needs sage.combinat
1.422388736882785488341547116024565825306879108991711829311892452916456747272565883
↪3312455412962072042
```

```
>>> from sage.all import *
>>> t = continued_fraction(words.ThueMorseWord([Integer(1),Integer(2)])); t    ␣
↪              # needs sage.combinat
[1; 2, 2, 1, 2, 1, 1, 2, 2, 1...]
>>> t.n(digits=Integer(100))                                                   ␣
↪     # needs sage.combinat
1.422388736882785488341547116024565825306879108991711829311892452916456747272565883
↪3312455412962072042
```

We check that comparisons work well:

```
sage: t > continued_fraction(1) and t < continued_fraction(3/2)                #␣
↪needs sage.combinat
True
sage: t < continued_fraction(1) or t > continued_fraction(2)                   #␣
↪needs sage.combinat
False
```

```
>>> from sage.all import *
>>> t > continued_fraction(Integer(1)) and t < continued_fraction(Integer(3)/
↪Integer(2))                  # needs sage.combinat
True
>>> t < continued_fraction(Integer(1)) or t > continued_fraction(Integer(2))   ␣
↪              # needs sage.combinat
False
```

Can also be called with a `value` option:

```
sage: def f(n):
....:     if n % 3 == 2: return 2*(n+1)//3
....:     return 1
sage: w = Word(f, alphabet=NN); w                                              #␣
↪needs sage.combinat
```

(continues on next page)

(continued from previous page)

```
word: 1,1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,12,1,1,14,1,1,16,1,1,18,1,1,20,1,1,22,1,
↪1,24,1,1,26,1,...
sage: cf = continued_fraction(w, value=e-1); cf                                   #␣
↪needs sage.combinat sage.symbolic
[1; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1...]
```

```
>>> from sage.all import *
>>> def f(n):
...     if n % Integer(3) == Integer(2): return Integer(2)*(n+Integer(1))//
↪Integer(3)
...     return Integer(1)
>>> w = Word(f, alphabet=NN); w                                                 #␣
↪needs sage.combinat
word: 1,1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,12,1,1,14,1,1,16,1,1,18,1,1,20,1,1,22,1,
↪1,24,1,1,26,1,...
>>> cf = continued_fraction(w, value=e-Integer(1)); cf                             ␣
↪     # needs sage.combinat sage.symbolic
[1; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1...]
```

In that case a small check is done on the input:

```
sage: cf = continued_fraction(w, value=pi)                                        #␣
↪needs sage.combinat sage.symbolic
Traceback (most recent call last):
...
ValueError: value evaluates to 3.141592653589794? while the continued
fraction evaluates to 1.718281828459046? in Real Interval Field
with 53 bits of precision.
```

```
>>> from sage.all import *
>>> cf = continued_fraction(w, value=pi)                                        #␣
↪needs sage.combinat sage.symbolic
Traceback (most recent call last):
...
ValueError: value evaluates to 3.141592653589794? while the continued
fraction evaluates to 1.718281828459046? in Real Interval Field
with 53 bits of precision.
```

length()

Return infinity.

EXAMPLES:

```
sage: w = words.FibonacciWord([3,13])                                         #␣
↪needs sage.combinat
sage: cf = continued_fraction(w)                                              #␣
↪needs sage.combinat
sage: cf.length()                                                             #␣
↪needs sage.combinat
+Infinity
```

```
>>> from sage.all import *
>>> w = words.FibonacciWord([Integer(3),Integer(13)])                     ␣
↪              # needs sage.combinat
>>> cf = continued_fraction(w)                                          #␣
↪needs sage.combinat
>>> cf.length()                                                         #␣
↪needs sage.combinat
+Infinity
```

quotient(*n*)

Return the n-th partial quotient of self.

INPUT:

- n – integer

EXAMPLES:

```
sage: # needs sage.combinat
sage: w = words.FibonacciWord([1,3])
sage: cf = continued_fraction(w)
sage: cf.quotient(0)
1
sage: cf.quotient(1)
3
sage: cf.quotient(2)
1
```

```
>>> from sage.all import *
>>> # needs sage.combinat
>>> w = words.FibonacciWord([Integer(1),Integer(3)])
>>> cf = continued_fraction(w)
>>> cf.quotient(Integer(0))
1
>>> cf.quotient(Integer(1))
3
>>> cf.quotient(Integer(2))
1
```

quotients()

Return the infinite list from which this continued fraction was built.

EXAMPLES:

```
sage: w = words.FibonacciWord([1,5])                                      #␣
↪needs sage.combinat
sage: cf = continued_fraction(w)                                          #␣
↪needs sage.combinat
sage: cf.quotients()                                                      #␣
↪needs sage.combinat
word: 1511515115115151151511511515115115151151...
```

```
>>> from sage.all import *
>>> w = words.FibonacciWord([Integer(1),Integer(5)])                      ␣
```

(continues on next page)

(continued from previous page)

```
↪                # needs sage.combinat
>>> cf = continued_fraction(w)                                              #␣
↪needs sage.combinat
>>> cf.quotients()                                                          #␣
↪needs sage.combinat
word: 1511515115115151151511515115151151...
```

value()

Return the value of `self`.

If this value was provided on initialization, just return this value otherwise return an element of the real lazy field.

EXAMPLES:

```
sage: def f(n):
....:     if n % 3 == 2: return 2*(n+1)//3
....:     return 1
sage: w = Word(f, alphabet=NN); w                                           #␣
↪needs sage.combinat
word: 1,1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,12,1,1,14,1,1,16,1,1,18,1,1,20,1,1,
↪22,1,1,24,1,1,26,1,...
sage: cf = continued_fraction(w, value=e-1); cf                             #␣
↪needs sage.combinat sage.symbolic
[1; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1...]
sage: cf.value()                                                            #␣
↪needs sage.combinat sage.symbolic
e - 1

sage: w = words.FibonacciWord([2,5])                                        #␣
↪needs sage.combinat
sage: cf = continued_fraction(w); cf                                        #␣
↪needs sage.combinat
[2; 5, 2, 2, 5, 2, 5, 2, 2, 5, 2, 2, 5, 2, 5, 2, 2, 5, 2, 5...]
sage: cf.value()                                                            #␣
↪needs sage.combinat
2.184951302409338?
```

```
>>> from sage.all import *
>>> def f(n):
...     if n % Integer(3) == Integer(2): return Integer(2)*(n+Integer(1))//
↪Integer(3)
...     return Integer(1)
>>> w = Word(f, alphabet=NN); w                                             #␣
↪needs sage.combinat
word: 1,1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,12,1,1,14,1,1,16,1,1,18,1,1,20,1,1,
↪22,1,1,24,1,1,26,1,...
>>> cf = continued_fraction(w, value=e-Integer(1)); cf                      ␣
↪      # needs sage.combinat sage.symbolic
[1; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1...]
>>> cf.value()                                                              #␣
↪needs sage.combinat sage.symbolic
e - 1
```

(continues on next page)

(continued from previous page)

```
>>> w = words.FibonacciWord([Integer(2),Integer(5)])                    ␣
↪               # needs sage.combinat
>>> cf = continued_fraction(w); cf                                    #␣
↪needs sage.combinat
[2; 5, 2, 2, 5, 2, 5, 2, 2, 5, 2, 2, 5, 2, 5, 2, 2, 5, 2, 5...]
>>> cf.value()                                                        #␣
↪needs sage.combinat
2.184951302409338?
```

class sage.rings.continued_fraction.**ContinuedFraction_periodic**(*x1*, *x2=None*, *check=True*)

Bases: *ContinuedFraction_base*

Continued fraction associated with rational or quadratic number.

A rational number has a finite continued fraction expansion (or ultimately 0). The one of a quadratic number, ie a number of the form $a + b\sqrt{D}$ with a and b rational, is ultimately periodic.

Note

This class stores a tuple `_x1` for the preperiod and a tuple `_x2` for the period. In the purely periodic case `_x1` is empty while in the rational case `_x2` is the tuple `(0,)`.

length()

Return the number of partial quotients of `self`.

EXAMPLES:

```
sage: continued_fraction(2/5).length()
3
sage: cf = continued_fraction([(0,1),(2,)]); cf
[0; 1, (2)*]
sage: cf.length()
+Infinity
```

```
>>> from sage.all import *
>>> continued_fraction(Integer(2)/Integer(5)).length()
3
>>> cf = continued_fraction([(Integer(0),Integer(1)),(Integer(2),)]); cf
[0; 1, (2)*]
>>> cf.length()
+Infinity
```

period()

Return the periodic part of `self`.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: K.<sqrt3> = QuadraticField(3)
sage: cf = continued_fraction(sqrt3); cf
[1; (1, 2)*]
```

(continues on next page)

(continued from previous page)

```
sage: cf.period()
(1, 2)
sage: for k in xsrange(2,40):
....:     if not k.is_square():
....:         s = QuadraticField(k).gen()
....:         cf = continued_fraction(s)
....:         print('%2d %d %s' % (k, len(cf.period()), cf))
 2 1 [1; (2)*]
 3 2 [1; (1, 2)*]
 5 1 [2; (4)*]
 6 2 [2; (2, 4)*]
 7 4 [2; (1, 1, 1, 4)*]
 8 2 [2; (1, 4)*]
10 1 [3; (6)*]
11 2 [3; (3, 6)*]
12 2 [3; (2, 6)*]
13 5 [3; (1, 1, 1, 1, 6)*]
14 4 [3; (1, 2, 1, 6)*]
...
35 2 [5; (1, 10)*]
37 1 [6; (12)*]
38 2 [6; (6, 12)*]
39 2 [6; (4, 12)*]
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = QuadraticField(Integer(3), names=('sqrt3',)); (sqrt3,) = K._first_
↪ngens(1)
>>> cf = continued_fraction(sqrt3); cf
[1; (1, 2)*]
>>> cf.period()
(1, 2)
>>> for k in xsrange(Integer(2),Integer(40)):
...     if not k.is_square():
...         s = QuadraticField(k).gen()
...         cf = continued_fraction(s)
...         print('%2d %d %s' % (k, len(cf.period()), cf))
 2 1 [1; (2)*]
 3 2 [1; (1, 2)*]
 5 1 [2; (4)*]
 6 2 [2; (2, 4)*]
 7 4 [2; (1, 1, 1, 4)*]
 8 2 [2; (1, 4)*]
10 1 [3; (6)*]
11 2 [3; (3, 6)*]
12 2 [3; (2, 6)*]
13 5 [3; (1, 1, 1, 1, 6)*]
14 4 [3; (1, 2, 1, 6)*]
...
35 2 [5; (1, 10)*]
37 1 [6; (12)*]
38 2 [6; (6, 12)*]
```

(continues on next page)

(continued from previous page)

```
39 2 [6; (4, 12)*]
```

period_length()

Return the number of partial quotients of the preperiodic part of `self`.

EXAMPLES:

```
sage: continued_fraction(2/5).period_length()
1
sage: cf = continued_fraction([(0,1),(2,)]); cf
[0; 1, (2)*]
sage: cf.period_length()
1
```

```
>>> from sage.all import *
>>> continued_fraction(Integer(2)/Integer(5)).period_length()
1
>>> cf = continued_fraction([(Integer(0),Integer(1)),(Integer(2),)]); cf
[0; 1, (2)*]
>>> cf.period_length()
1
```

preperiod()

Return the preperiodic part of `self`.

EXAMPLES:

```
sage: # needs sage.rings.number_field
sage: K.<sqrt3> = QuadraticField(3)
sage: cf = continued_fraction(sqrt3); cf
[1; (1, 2)*]
sage: cf.preperiod()
(1,)
sage: cf = continued_fraction(sqrt3/7); cf
[0; 4, (24, 8)*]
sage: cf.preperiod()
(0, 4)
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> K = QuadraticField(Integer(3), names=('sqrt3',)); (sqrt3,) = K._first_
↪ngens(1)
>>> cf = continued_fraction(sqrt3); cf
[1; (1, 2)*]
>>> cf.preperiod()
(1,)
>>> cf = continued_fraction(sqrt3/Integer(7)); cf
[0; 4, (24, 8)*]
>>> cf.preperiod()
(0, 4)
```

preperiod_length()

Return the number of partial quotients of the preperiodic part of `self`.

EXAMPLES:

```
sage: continued_fraction(2/5).preperiod_length()
3
sage: cf = continued_fraction([(0,1),(2,)]); cf
[0; 1, (2)*]
sage: cf.preperiod_length()
2
```

```
>>> from sage.all import *
>>> continued_fraction(Integer(2)/Integer(5)).preperiod_length()
3
>>> cf = continued_fraction([(Integer(0),Integer(1)),(Integer(2),)]); cf
[0; 1, (2)*]
>>> cf.preperiod_length()
2
```

quotient(*n*)

Return the n-th partial quotient of `self`.

EXAMPLES:

```
sage: cf = continued_fraction([(12,5),(1,3)])
sage: [cf.quotient(i) for i in range(10)]
[12, 5, 1, 3, 1, 3, 1, 3, 1, 3]
```

```
>>> from sage.all import *
>>> cf = continued_fraction([(Integer(12),Integer(5)),(Integer(1),
↪Integer(3))])
>>> [cf.quotient(i) for i in range(Integer(10))]
[12, 5, 1, 3, 1, 3, 1, 3, 1, 3]
```

value()

Return the value of `self` as a quadratic number (with square free discriminant).

EXAMPLES:

Some purely periodic examples:

```
sage: cf = continued_fraction([(),(2,)]); cf
[(2)*]
sage: v = cf.value(); v                                                   #␣
↪needs sage.rings.number_field
sqrt2 + 1
sage: v.continued_fraction()                                              #␣
↪needs sage.rings.number_field
[(2)*]

sage: cf = continued_fraction([(),(1,2)]); cf
[(1, 2)*]
sage: v = cf.value(); v                                                   #␣
↪needs sage.rings.number_field
1/2*sqrt3 + 1/2
sage: v.continued_fraction()                                              #␣
```

(continues on next page)

(continued from previous page)

```
↪needs sage.rings.number_field
[(1, 2)*]
```

```
>>> from sage.all import *
>>> cf = continued_fraction([(),(Integer(2),)]); cf
[(2)*]
>>> v = cf.value(); v                                                    #␣
↪needs sage.rings.number_field
sqrt2 + 1
>>> v.continued_fraction()                                               #␣
↪needs sage.rings.number_field
[(2)*]

>>> cf = continued_fraction([(),(Integer(1),Integer(2))]); cf
[(1, 2)*]
>>> v = cf.value(); v                                                    #␣
↪needs sage.rings.number_field
1/2*sqrt3 + 1/2
>>> v.continued_fraction()                                               #␣
↪needs sage.rings.number_field
[(1, 2)*]
```

The number `sqrt3` that appear above is actually internal to the continued fraction. In order to be access it from the console:

```
sage: cf.value().parent().inject_variables()                               #␣
↪needs sage.rings.number_field
Defining sqrt3
sage: sqrt3                                                                #␣
↪needs sage.rings.number_field
sqrt3
sage: ((sqrt3+1)/2).continued_fraction()                                   #␣
↪needs sage.rings.number_field
[(1, 2)*]
```

```
>>> from sage.all import *
>>> cf.value().parent().inject_variables()                               #␣
↪needs sage.rings.number_field
Defining sqrt3
>>> sqrt3                                                                #␣
↪needs sage.rings.number_field
sqrt3
>>> ((sqrt3+Integer(1))/Integer(2)).continued_fraction()                    ␣
↪               # needs sage.rings.number_field
[(1, 2)*]
```

Some ultimately periodic but non periodic examples:

```
sage: cf = continued_fraction([(1,),(2,)]); cf
[1; (2)*]
sage: v = cf.value(); v                                                    #␣
↪needs sage.rings.number_field
```

(continues on next page)

(continued from previous page)

```
sqrt2
sage: v.continued_fraction()                                                   #␣
↪needs sage.rings.number_field
[1; (2)*]

sage: cf = continued_fraction([(1,3),(1,2)]); cf
[1; 3, (1, 2)*]
sage: v = cf.value(); v                                                        #␣
↪needs sage.rings.number_field
-sqrt3 + 3
sage: v.continued_fraction()                                                   #␣
↪needs sage.rings.number_field
[1; 3, (1, 2)*]

sage: cf = continued_fraction([(-5,18), (1,3,1,5)])
sage: cf.value().continued_fraction() == cf                                    #␣
↪needs sage.rings.number_field
True
sage: cf = continued_fraction([(-1,),(1,)])
sage: cf.value().continued_fraction() == cf                                    #␣
↪needs sage.rings.number_field
True
```

```
>>> from sage.all import *
>>> cf = continued_fraction([(Integer(1),),(Integer(2),)]); cf
[1; (2)*]
>>> v = cf.value(); v                                                        #␣
↪needs sage.rings.number_field
sqrt2
>>> v.continued_fraction()                                                   #␣
↪needs sage.rings.number_field
[1; (2)*]

>>> cf = continued_fraction([(Integer(1),Integer(3)),(Integer(1),
↪Integer(2))]); cf
[1; 3, (1, 2)*]
>>> v = cf.value(); v                                                        #␣
↪needs sage.rings.number_field
-sqrt3 + 3
>>> v.continued_fraction()                                                   #␣
↪needs sage.rings.number_field
[1; 3, (1, 2)*]

>>> cf = continued_fraction([(-Integer(5),Integer(18)), (Integer(1),
↪Integer(3),Integer(1),Integer(5))])
>>> cf.value().continued_fraction() == cf                                    #␣
↪needs sage.rings.number_field
True
>>> cf = continued_fraction([(-Integer(1),),(Integer(1),)])
>>> cf.value().continued_fraction() == cf                                    #␣
↪needs sage.rings.number_field
True
```

class sage.rings.continued_fraction.**ContinuedFraction_real**(*x*)

Bases: *ContinuedFraction_base*

Continued fraction of a real (exact) number.

This class simply wraps a real number into an attribute (that can be accessed through the method *value()*). The number is assumed to be irrational.

EXAMPLES:

```
sage: cf = continued_fraction(pi); cf                                    #␣
↪needs sage.symbolic
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: cf.value()                                                         #␣
↪needs sage.symbolic
pi

sage: cf = continued_fraction(e); cf                                     #␣
↪needs sage.symbolic
[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, ...]
sage: cf.value()                                                         #␣
↪needs sage.symbolic
e
```

```
>>> from sage.all import *
>>> cf = continued_fraction(pi); cf                                    #␣
↪needs sage.symbolic
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
>>> cf.value()                                                         #␣
↪needs sage.symbolic
pi

>>> cf = continued_fraction(e); cf                                     #␣
↪needs sage.symbolic
[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, ...]
>>> cf.value()                                                         #␣
↪needs sage.symbolic
e
```

length()

Return infinity.

EXAMPLES:

```
sage: continued_fraction(pi).length()                                    #␣
↪needs sage.symbolic
+Infinity
```

```
>>> from sage.all import *
>>> continued_fraction(pi).length()                                    #␣
↪needs sage.symbolic
+Infinity
```

quotient(*n*)

Return the n-th quotient of self.

EXAMPLES:

```
sage: # needs sage.symbolic
sage: cf = continued_fraction(pi)
sage: cf.quotient(27)
13
sage: cf.quotient(2552)
152
sage: cf.quotient(10000)            # long time
5
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> cf = continued_fraction(pi)
>>> cf.quotient(Integer(27))
13
>>> cf.quotient(Integer(2552))
152
>>> cf.quotient(Integer(10000))            # long time
5
```

The algorithm is not efficient with element of the symbolic ring and, if possible, one can always prefer number fields elements. The reason is that, given a symbolic element `x`, there is no automatic way to evaluate in `RIF` an expression of the form `(a*x+b)/(c*x+d)` where both the numerator and the denominator are extremely small:

```
sage: # needs sage.symbolic
sage: a1 = pi
sage: c1 = continued_fraction(a1)
sage: p0 = c1.numerator(12); q0 = c1.denominator(12)
sage: p1 = c1.numerator(13); q1 = c1.denominator(13)
sage: num = (q0*a1 - p0); num.n()
1.49011611938477e-8
sage: den = (q1*a1 - p1); den.n()
-2.98023223876953e-8
sage: a1 = -num/den
sage: RIF(a1)
[-infinity .. +infinity]
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> a1 = pi
>>> c1 = continued_fraction(a1)
>>> p0 = c1.numerator(Integer(12)); q0 = c1.denominator(Integer(12))
>>> p1 = c1.numerator(Integer(13)); q1 = c1.denominator(Integer(13))
>>> num = (q0*a1 - p0); num.n()
1.49011611938477e-8
>>> den = (q1*a1 - p1); den.n()
-2.98023223876953e-8
>>> a1 = -num/den
>>> RIF(a1)
[-infinity .. +infinity]
```

The same computation with an element of a number field instead of `pi` gives a very satisfactory answer:

```
sage: # needs sage.rings.number_field
sage: x = polygen(ZZ, 'x')
sage: K.<a2> = NumberField(x^3 - 2, embedding=1.25)
sage: c2 = continued_fraction(a2)
sage: p0 = c2.numerator(111); q0 = c2.denominator(111)
sage: p1 = c2.numerator(112); q1 = c2.denominator(112)
sage: num = (q0*a2 - p0); num.n()
-4.56719261665907e46
sage: den = (q1*a2 - p1); den.n()
-3.65375409332726e47
sage: a2 = -num/den
sage: b2 = RIF(a2); b2
1.002685823312715?
sage: b2.absolute_diameter()
8.88178419700125e-16
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> x = polygen(ZZ, 'x')
>>> K = NumberField(x**Integer(3) - Integer(2), embedding=RealNumber('1.25'),␣
↪names=('a2',)); (a2,) = K._first_ngens(1)
>>> c2 = continued_fraction(a2)
>>> p0 = c2.numerator(Integer(111)); q0 = c2.denominator(Integer(111))
>>> p1 = c2.numerator(Integer(112)); q1 = c2.denominator(Integer(112))
>>> num = (q0*a2 - p0); num.n()
-4.56719261665907e46
>>> den = (q1*a2 - p1); den.n()
-3.65375409332726e47
>>> a2 = -num/den
>>> b2 = RIF(a2); b2
1.002685823312715?
>>> b2.absolute_diameter()
8.88178419700125e-16
```

The consequence is that the precision needed with `c1` grows when we compute larger and larger partial quotients:

```
sage: # needs sage.symbolic
sage: c1.quotient(100)
2
sage: c1._xa.parent()
Real Interval Field with 353 bits of precision
sage: c1.quotient(200)
3
sage: c1._xa.parent()
Real Interval Field with 753 bits of precision
sage: c1.quotient(300)
5
sage: c1._xa.parent()
Real Interval Field with 1053 bits of precision

sage: # needs sage.rings.number_field
sage: c2.quotient(200)
```

(continues on next page)

(continued from previous page)

```
6
sage: c2._xa.parent()
Real Interval Field with 53 bits of precision
sage: c2.quotient(500)
1
sage: c2._xa.parent()
Real Interval Field with 53 bits of precision
sage: c2.quotient(1000)
1
sage: c2._xa.parent()
Real Interval Field with 53 bits of precision
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> c1.quotient(Integer(100))
2
>>> c1._xa.parent()
Real Interval Field with 353 bits of precision
>>> c1.quotient(Integer(200))
3
>>> c1._xa.parent()
Real Interval Field with 753 bits of precision
>>> c1.quotient(Integer(300))
5
>>> c1._xa.parent()
Real Interval Field with 1053 bits of precision

>>> # needs sage.rings.number_field
>>> c2.quotient(Integer(200))
6
>>> c2._xa.parent()
Real Interval Field with 53 bits of precision
>>> c2.quotient(Integer(500))
1
>>> c2._xa.parent()
Real Interval Field with 53 bits of precision
>>> c2.quotient(Integer(1000))
1
>>> c2._xa.parent()
Real Interval Field with 53 bits of precision
```

value()

Return the value of `self` (the number from which it was built).

EXAMPLES:

```
sage: cf = continued_fraction(e)                                        #␣
↪needs sage.symbolic
sage: cf.value()                                                        #␣
↪needs sage.symbolic
e
```

```
>>> from sage.all import *
>>> cf = continued_fraction(e)                                              #␣
↪needs sage.symbolic
>>> cf.value()                                                              #␣
↪needs sage.symbolic
e
```

sage.rings.continued_fraction.**check_and_reduce_pair**(*x1*, *x2=None*)

There are often two ways to represent a given continued fraction. This function makes it canonical.

In the very special case of the number 0 we return the pair `((0,),(0,))`.

sage.rings.continued_fraction.**continued_fraction**(*x*, *value=None*)

Return the continued fraction of x.

INPUT:

- `x` – a number or a list of partial quotients (for finite development) or two list of partial quotients (preperiod and period for ultimately periodic development)

EXAMPLES:

A finite continued fraction may be initialized by a number or by its list of partial quotients:

```
sage: continued_fraction(12/571)
[0; 47, 1, 1, 2, 2]
sage: continued_fraction([3,2,1,4])
[3; 2, 1, 4]
```

```
>>> from sage.all import *
>>> continued_fraction(Integer(12)/Integer(571))
[0; 47, 1, 1, 2, 2]
>>> continued_fraction([Integer(3),Integer(2),Integer(1),Integer(4)])
[3; 2, 1, 4]
```

It can be called with elements defined from symbolic values, in which case the partial quotients are evaluated in a lazy way:

```
sage: c = continued_fraction(golden_ratio); c                                #␣
↪needs sage.symbolic
[1; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...]
sage: c.convergent(12)                                                       #␣
↪needs sage.symbolic
377/233
sage: fibonacci(14)/fibonacci(13)                                            #␣
↪needs sage.libs.pari
377/233

sage: # needs sage.symbolic
sage: continued_fraction(pi)
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: a = c.convergent(3); a
355/113
```

(continues on next page)

(continued from previous page)

```
sage: a.n()
3.14159292035398
sage: pi.n()
3.14159265358979

sage: # needs sage.symbolic
sage: continued_fraction(sqrt(2))
[1; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, ...]
sage: continued_fraction(tan(1))
[1; 1, 1, 3, 1, 5, 1, 7, 1, 9, 1, 11, 1, 13, 1, 15, 1, 17, 1, 19, ...]
sage: continued_fraction(tanh(1))
[0; 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, ...]
sage: continued_fraction(e)
[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, ...]
```

```
>>> from sage.all import *
>>> c = continued_fraction(golden_ratio); c                                 #␣
↪needs sage.symbolic
[1; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...]
>>> c.convergent(Integer(12))                                                ␣
↪     # needs sage.symbolic
377/233
>>> fibonacci(Integer(14))/fibonacci(Integer(13))                            ␣
↪              # needs sage.libs.pari
377/233

>>> # needs sage.symbolic
>>> continued_fraction(pi)
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
>>> c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
>>> a = c.convergent(Integer(3)); a
355/113
>>> a.n()
3.14159292035398
>>> pi.n()
3.14159265358979

>>> # needs sage.symbolic
>>> continued_fraction(sqrt(Integer(2)))
[1; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, ...]
>>> continued_fraction(tan(Integer(1)))
[1; 1, 1, 3, 1, 5, 1, 7, 1, 9, 1, 11, 1, 13, 1, 15, 1, 17, 1, 19, ...]
>>> continued_fraction(tanh(Integer(1)))
[0; 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, ...]
>>> continued_fraction(e)
[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, ...]
```

If you want to play with quadratic numbers (such as `golden_ratio` and `sqrt(2)` above), it is much more convenient to use number fields as follows since preperiods and periods are computed:

```
sage: # needs sage.rings.number_field
```

(continues on next page)

(continued from previous page)

```
sage: x = polygen(ZZ, 'x')
sage: K.<sqrt5> = NumberField(x^2 - 5, embedding=2.23)
sage: my_golden_ratio = (1 + sqrt5)/2
sage: cf = continued_fraction((1+sqrt5)/2); cf
[(1)*]
sage: cf.convergent(12)
377/233
sage: cf.period()
(1,)
sage: cf = continued_fraction(2/3+sqrt5/5); cf
[1; 8, (1, 3, 1, 1, 3, 9)*]
sage: cf.preperiod()
(1, 8)
sage: cf.period()
(1, 3, 1, 1, 3, 9)

sage: # needs sage.rings.number_field
sage: L.<sqrt2> = NumberField(x^2 - 2, embedding=1.41)
sage: cf = continued_fraction(sqrt2); cf
[1; (2)*]
sage: cf.period()
(2,)
sage: cf = continued_fraction(sqrt2/3); cf
[0; 2, (8, 4)*]
sage: cf.period()
(8, 4)
```

```
>>> from sage.all import *
>>> # needs sage.rings.number_field
>>> x = polygen(ZZ, 'x')
>>> K = NumberField(x**Integer(2) - Integer(5), embedding=RealNumber('2.23'),␣
↪names=('sqrt5',)); (sqrt5,) = K._first_ngens(1)
>>> my_golden_ratio = (Integer(1) + sqrt5)/Integer(2)
>>> cf = continued_fraction((Integer(1)+sqrt5)/Integer(2)); cf
[(1)*]
>>> cf.convergent(Integer(12))
377/233
>>> cf.period()
(1,)
>>> cf = continued_fraction(Integer(2)/Integer(3)+sqrt5/Integer(5)); cf
[1; 8, (1, 3, 1, 1, 3, 9)*]
>>> cf.preperiod()
(1, 8)
>>> cf.period()
(1, 3, 1, 1, 3, 9)

>>> # needs sage.rings.number_field
>>> L = NumberField(x**Integer(2) - Integer(2), embedding=RealNumber('1.41'),␣
↪names=('sqrt2',)); (sqrt2,) = L._first_ngens(1)
>>> cf = continued_fraction(sqrt2); cf
[1; (2)*]
>>> cf.period()
```

(continues on next page)

(continued from previous page)

```
(2,)
>>> cf = continued_fraction(sqrt2/Integer(3)); cf
[0; 2, (8, 4)*]
>>> cf.period()
(8, 4)
```

It is also possible to go the other way around, build a ultimately periodic continued fraction from its preperiod and its period and get its value back:

```
sage: cf = continued_fraction([(1,1), (2,8)]); cf
[1; 1, (2, 8)*]
sage: cf.value()                                                          #␣
↪needs sage.rings.number_field
2/11*sqrt5 + 14/11
```

```
>>> from sage.all import *
>>> cf = continued_fraction([(Integer(1),Integer(1)), (Integer(2),Integer(8))]);␣
↪cf
[1; 1, (2, 8)*]
>>> cf.value()                                                          #␣
↪needs sage.rings.number_field
2/11*sqrt5 + 14/11
```

It is possible to deal with higher degree number fields but in that case the continued fraction expansion is known to be aperiodic:

```
sage: K.<a> = NumberField(x^3 - 2, embedding=1.25)                        #␣
↪needs sage.rings.number_field
sage: cf = continued_fraction(a); cf                                      #␣
↪needs sage.rings.number_field
[1; 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, ...]
```

```
>>> from sage.all import *
>>> K = NumberField(x**Integer(3) - Integer(2), embedding=RealNumber('1.25'),␣
↪names=('a',)); (a,) = K._first_ngens(1)# needs sage.rings.number_field
>>> cf = continued_fraction(a); cf                                      #␣
↪needs sage.rings.number_field
[1; 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, ...]
```

Note that initial rounding can result in incorrect trailing partial quotients:

```
sage: continued_fraction(RealField(39)(e))                                #␣
↪needs sage.symbolic
[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 2]
```

```
>>> from sage.all import *
>>> continued_fraction(RealField(Integer(39))(e))                         ␣
↪    # needs sage.symbolic
[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 2]
```

Note the value returned for floating point number is the continued fraction associated to the rational number you obtain with a conversion:

```
sage: for _ in range(10):
....:     x = RR.random_element()
....:     cff = continued_fraction(x)
....:     cfe = QQ(x).continued_fraction()
....:     assert cff == cfe, "%s %s %s"%(x,cff,cfe)
```

```
>>> from sage.all import *
>>> for _ in range(Integer(10)):
...     x = RR.random_element()
...     cff = continued_fraction(x)
...     cfe = QQ(x).continued_fraction()
...     assert cff == cfe, "%s %s %s"%(x,cff,cfe)
```

sage.rings.continued_fraction.**continued_fraction_list**(*x*, *type='std'*, *partial_convergents=False*, *bits=None*, *nterms=None*)

Return the (finite) continued fraction of x as a list.

The continued fraction expansion of x are the coefficients a_i in

$$x = a_0 + 1/(a_1 + 1/(...))$$

with a_0 integer and a_1, ... positive integers. The Hirzebruch-Jung continued fraction is the one for which the $+$ signs are replaced with $-$ signs

$$x = a_0 - 1/(a_1 - 1/(...))$$

See also

continued_fraction()

INPUT:

- x – exact rational or floating-point number; the number to compute the continued fraction of
- type – either 'std' (default) for standard continued fractions or 'hj' for Hirzebruch-Jung ones
- partial_convergents – boolean; whether to return the partial convergents
- bits – an optional integer that specify a precision for the real interval field that is used internally
- nterms – integer; the upper bound on the number of terms in the continued fraction expansion to return

OUTPUT:

A lits of integers, the coefficients in the continued fraction expansion of x. If partial_convergents is set to True, then return a pair containing the coefficient list and the partial convergents list is returned.

EXAMPLES:

```
sage: continued_fraction_list(45/19)
[2, 2, 1, 2, 2]
sage: 2 + 1/(2 + 1/(1 + 1/(2 + 1/2)))
45/19

sage: continued_fraction_list(45/19, type='hj')
```

(continues on next page)

(continued from previous page)

```
[3, 2, 3, 2, 3]
sage: 3 - 1/(2 - 1/(3 - 1/(2 - 1/3)))
45/19
```

```
>>> from sage.all import *
>>> continued_fraction_list(Integer(45)/Integer(19))
[2, 2, 1, 2, 2]
>>> Integer(2) + Integer(1)/(Integer(2) + Integer(1)/(Integer(1) +
↪Integer(1)/(Integer(2) + Integer(1)/Integer(2))))
45/19

>>> continued_fraction_list(Integer(45)/Integer(19), type='hj')
[3, 2, 3, 2, 3]
>>> Integer(3) - Integer(1)/(Integer(2) - Integer(1)/(Integer(3) -
↪Integer(1)/(Integer(2) - Integer(1)/Integer(3))))
45/19
```

Specifying `bits` or `nterms` modify the length of the output:

```
sage: # needs sage.symbolic
sage: continued_fraction_list(e, bits=20)
[2, 1, 2, 1, 1, 4, 2]
sage: continued_fraction_list(sqrt(2) + sqrt(3), bits=30)
[3, 6, 1, 5, 7, 2]
sage: continued_fraction_list(pi, bits=53)
[3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14]
sage: continued_fraction_list(log(3/2), nterms=15)
[0, 2, 2, 6, 1, 11, 2, 1, 2, 2, 1, 4, 3, 1, 1]
sage: continued_fraction_list(tan(sqrt(pi)), nterms=20)
[-5, 9, 4, 1, 1, 1, 1, 1, 1, 5, 1, 1, 1, 1, 1, 2, 4, 3, 1, 63]
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> continued_fraction_list(e, bits=Integer(20))
[2, 1, 2, 1, 1, 4, 2]
>>> continued_fraction_list(sqrt(Integer(2)) + sqrt(Integer(3)), bits=Integer(30))
[3, 6, 1, 5, 7, 2]
>>> continued_fraction_list(pi, bits=Integer(53))
[3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14]
>>> continued_fraction_list(log(Integer(3)/Integer(2)), nterms=Integer(15))
[0, 2, 2, 6, 1, 11, 2, 1, 2, 2, 1, 4, 3, 1, 1]
>>> continued_fraction_list(tan(sqrt(pi)), nterms=Integer(20))
[-5, 9, 4, 1, 1, 1, 1, 1, 1, 5, 1, 1, 1, 1, 1, 2, 4, 3, 1, 63]
```

When the continued fraction is infinite (ie `x` is an irrational number) and the parameters `bits` and `nterms` are not specified then a warning is raised:

```
sage: continued_fraction_list(sqrt(2))                                  #
↪needs sage.symbolic
doctest:...: UserWarning: the continued fraction of sqrt(2) seems infinite,
return only the first 20 terms
[1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]
```

(continues on next page)

(continued from previous page)

```
sage: continued_fraction_list(sqrt(4/19))                                    #␣
↪needs sage.symbolic
doctest:...: UserWarning: the continued fraction of 2*sqrt(1/19) seems infinite,
return only the first 20 terms
[0, 2, 5, 1, 1, 2, 1, 16, 1, 2, 1, 1, 5, 4, 5, 1, 1, 2, 1, 16]
```

```
>>> from sage.all import *
>>> continued_fraction_list(sqrt(Integer(2)))                                 ␣
↪     # needs sage.symbolic
doctest:...: UserWarning: the continued fraction of sqrt(2) seems infinite,
return only the first 20 terms
[1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]
>>> continued_fraction_list(sqrt(Integer(4)/Integer(19)))                     ␣
↪              # needs sage.symbolic
doctest:...: UserWarning: the continued fraction of 2*sqrt(1/19) seems infinite,
return only the first 20 terms
[0, 2, 5, 1, 1, 2, 1, 16, 1, 2, 1, 1, 5, 4, 5, 1, 1, 2, 1, 16]
```

An examples with the list of partial convergents:

```
sage: continued_fraction_list(RR(pi), partial_convergents=True)              #␣
↪needs sage.symbolic
([3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 3],
 [(3, 1),
  (22, 7),
  (333, 106),
  (355, 113),
  (103993, 33102),
  (104348, 33215),
  (208341, 66317),
  (312689, 99532),
  (833719, 265381),
  (1146408, 364913),
  (4272943, 1360120),
  (5419351, 1725033),
  (80143857, 25510582),
  (245850922, 78256779)])
```

```
>>> from sage.all import *
>>> continued_fraction_list(RR(pi), partial_convergents=True)              #␣
↪needs sage.symbolic
([3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 3],
 [(3, 1),
  (22, 7),
  (333, 106),
  (355, 113),
  (103993, 33102),
  (104348, 33215),
  (208341, 66317),
  (312689, 99532),
  (833719, 265381),
  (1146408, 364913),
```

(continues on next page)

(continued from previous page)

```
 (4272943, 1360120),
 (5419351, 1725033),
 (80143857, 25510582),
 (245850922, 78256779)])
```

sage.rings.continued_fraction.**convergents**(*x*)

Return the (partial) convergents of the number x.

EXAMPLES:

```
sage: from sage.rings.continued_fraction import convergents
sage: convergents(143/255)
[0, 1, 1/2, 4/7, 5/9, 9/16, 14/25, 23/41, 60/107, 143/255]
```

```
>>> from sage.all import *
>>> from sage.rings.continued_fraction import convergents
>>> convergents(Integer(143)/Integer(255))
[0, 1, 1/2, 4/7, 5/9, 9/16, 14/25, 23/41, 60/107, 143/255]
```

sage.rings.continued_fraction.**last_two_convergents**(*x*)

Given the list x that consists of numbers, return the two last convergents $p_{n-1}, q_{n-1}, p_n, q_n$.

This function is principally used to compute the value of a ultimately periodic continued fraction.

OUTPUT: a 4-tuple of Sage integers

EXAMPLES:

```
sage: from sage.rings.continued_fraction import last_two_convergents
sage: last_two_convergents([])
(0, 1, 1, 0)
sage: last_two_convergents([0])
(1, 0, 0, 1)
sage: last_two_convergents([-1,1,3,2])
(-1, 4, -2, 9)
```

```
>>> from sage.all import *
>>> from sage.rings.continued_fraction import last_two_convergents
>>> last_two_convergents([])
(0, 1, 1, 0)
>>> last_two_convergents([Integer(0)])
(1, 0, 0, 1)
>>> last_two_convergents([-Integer(1),Integer(1),Integer(3),Integer(2)])
(-1, 4, -2, 9)
```

sage.rings.continued_fraction.**rat_interval_cf_list**(*r1*, *r2*)

Return the common prefix of the rationals r1 and r2 seen as continued fractions.

OUTPUT: list of Sage integers

EXAMPLES:

```
sage: from sage.rings.continued_fraction import rat_interval_cf_list
sage: rat_interval_cf_list(257/113, 5224/2297)
```

(continues on next page)

(continued from previous page)

```
[2, 3, 1, 1, 1, 4]
sage: for prec in range(10,54):                                          #␣
↪needs sage.rings.real_interval_field
....:     R = RealIntervalField(prec)
....:     for _ in range(100):
....:         x = R.random_element() * R.random_element() + R.random_element() /␣
↪100
....:         l = x.lower().exact_rational()
....:         u = x.upper().exact_rational()
....:         if l.floor() != u.floor():
....:             continue
....:         cf = rat_interval_cf_list(l,u)
....:         a = continued_fraction(cf).value()
....:         b = continued_fraction(cf+[1]).value()
....:         if a > b:
....:             a,b = b,a
....:         assert a <= l
....:         assert b >= u
```

```
>>> from sage.all import *
>>> from sage.rings.continued_fraction import rat_interval_cf_list
>>> rat_interval_cf_list(Integer(257)/Integer(113), Integer(5224)/Integer(2297))
[2, 3, 1, 1, 1, 4]
>>> for prec in range(Integer(10),Integer(54)):                           ␣
↪              # needs sage.rings.real_interval_field
...     R = RealIntervalField(prec)
...     for _ in range(Integer(100)):
...         x = R.random_element() * R.random_element() + R.random_element() /␣
↪Integer(100)
...         l = x.lower().exact_rational()
...         u = x.upper().exact_rational()
...         if l.floor() != u.floor():
...             continue
...         cf = rat_interval_cf_list(l,u)
...         a = continued_fraction(cf).value()
...         b = continued_fraction(cf+[Integer(1)]).value()
...         if a > b:
...             a,b = b,a
...         assert a <= l
...         assert b >= u
```

CHAPTER

TWO

INDICES AND TABLES

- Index
- Module Index
- Search Page

PYTHON MODULE INDEX

r

INDEX

Q

R

S

V